EE 562a

Homework 5

Due Wednesday July 8, 2009

Work the following 5 problems.

Problem 1. Let $l_2 = l_2[0, \infty)$ be the space of all sequences $x = (x_1, x_2, \ldots)$ of complex numbers such that

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty.$$

Define

$$< x, y >= \sum_{i=1}^{\infty} x_i y_i^*.$$

Then l_2 is a Hilbert space.

Now let H be any Hilbert space. Then $M \subseteq H$ is a called a *linear manifold* in H if for all $x, y \in M$ and all complex numbers α we have $x + y \in M$ and $\alpha x \in M$. (Note: When H is finite dimensional then a linear manifold is a vector subspace.)

Let

$$M = \{x \in l_2 : x = (x_1, x_2, \ldots) \text{ and all but finitely many } x_k \text{ equal zero}\}.$$

Show that M is a linear manifold in H that is not closed.

Hint: Construct a Cauchy sequence in M that does not converge to an element in M.

Problem 2. If X and Y are any two random variables, find the best estimator (or predictor) of Y given X for mean absolute error, i.e., find $g(X)$ so that $E[|Y - g(X)|]$ is minimal.

Problem 3. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi F t + \theta).$$

Here the amplitude A is a constant but we do *not* know its value. We do know that the frequency F is 200 Hz and the phase θ is $\pi/4$. We can follow the steps below to estimate the value of A. Assume for purposes of calculation that the value of A is 10, i.e., use $A = 10$ in the above signal in your calculations.

S1. Multiply $r_a(t)$ by $x(t)$, where $x(t) = \cos(2\pi 200t + \pi/4)$. Call the result $y(t)$.

S2. Integrate $y(t)$ from 0 to T and multiply the result by $2/T$. The result is your estimate of A.

a. Follow the 2 steps above and estimate A using $T = 3,\ 7,\ 17,\ 27,\ 127$ msec.

b. Explain why following the 2 steps above will give the exact answer for A as $T \to \infty$.

c. Determine (analytically) the *finite* values of T that will make your estimate for A exact and using the smallest such T follow the two steps above again to estimate A.

Problem 4. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi 200t + \theta)$$

and sample it at 500 Hz to get the digital signal

$$r(n) = A\cos(0.8\pi n + \theta).$$

Here the amplitude A is a constant but we do *not* know its value. Furthermore, we do *not* know the θ phase value. We can follow the steps below to estimate the value of A. Assume for purposes of calculation that the value of A is 10 and $\theta = \pi/4$, i.e., use $A = 10$ and $\theta = \pi/4$ in the above signal in your calculations. You can work this problem using Matlab or Excel or some other software except for the analytic part.

S1. Multiply $r(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.8\pi n)$ and $x_2(n) = \sin(0.8\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

S2. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some N) and take the average of each (divide by N) and then multiply the averages by 2. Call the results z_1 and z_2, respectively.

S3. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of A.

a. Follow the 3 steps above and estimate A using $N = 5, 11, 19, 29$.

b. Explain why following the 3 steps above will give the exact answer for A as $N \to \infty$.

d. Determine (analytically) the *finite* values of N that will make your estimate for A exact and using the smallest such N follow the 3 steps above again to estimate A.

Problem 5. Suppose we receive the quantized digital signal

$$r_q(n) = \text{Round}\,[A\cos(0.8\pi n + \theta)]$$

where 'Round' means the samples are rounded to the nearest integer (in a real analog to digital (A/D) converter this rounding would map to a certain level of the A/D depending on the number of levels of the A/D but it is sufficient for this problem to just assume the rounding results in the integer obtained by the usual rounding operation). The amplitude A is a constant but we do not know its value. Furthermore, we do not know that the phase is $\theta = \pi/8$. We can follow the steps below to estimate the value of A. [For purposes of calculation let A actually have the value 5.]

S1. Multiply $r_q(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.80\pi n)$ and $x_2(n) = \sin(0.80\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

S2. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some N) and take the average of each (divide by N) and then multiply the averages by 2. Call the results z_1 and z_2, respectively.

S3. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of A.

a. Follow the 3 steps above and estimate A using $N = 5$.

b. Repeat (a) for $N = 10$.

c. Making N multiples of 5 estimates A exactly if the input samples were not quantized. Based on your answers to parts (a) and (b) what would your estimate for A be if N is 5000. Explain why the estimate is not becoming exact even for very large N. If you were designing the system what could you do to avoid this problem?