EE 562a

Homework 4

Due Wednesday June 17, 2009

Work the following 5 problems.

Problem 1. Design a correlation detector for deciding whether H_0 or H_1 is true where

$$H_i: \ \mathbf{X}(u) = \mathbf{S}_i + \mathbf{N}(u), \quad i = 0, 1$$

and

$$\mathbf{S}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{S}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$E[\mathbf{N}(u)] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{K_N} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Indicate the decision regions on a graph.

Solution.

$$\mathbf{K_N}^{-1} = \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Say "H_0 True" when

$$\begin{aligned}
\left\langle \mathbf{x} - \mu_{\mathbf{N}}, \mathbf{K_N}^{-1}(\mathbf{s_1} - \mathbf{s_0}) \right\rangle &< \frac{\frac{3}{8} - \frac{3}{8}}{2} \\
\left(\frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)^T \left(\mathbf{x} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &< 0 \\
\frac{1}{8}[1 \ \ 3]\left(\mathbf{x} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &< 0 \\
x(u,1) + 3x(u,2) &< 4.
\end{aligned}$$

And say "H_1 true" otherwise.

The decision regions are shown in figure 1 below.

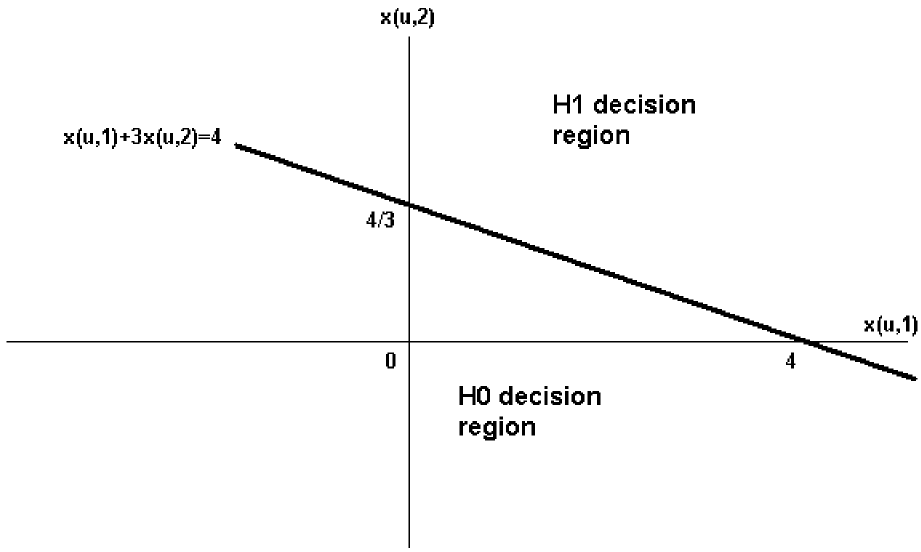

Figure 1: Decision region for problem 1

Problem 2. We are given an observation of X and we must decide between two hypotheses:

$$H_0: \quad X = N$$
$$H_1: \quad X = S + N$$

where N has density

$$f_N(x) = \frac{3}{2}e^{-3|x|}, \quad x \in (-\infty, \infty)$$

and $S = 4.5$.

a. Find a threshold T so that the probability of type I error is 10^{-5}. Note: You can assume S will always be greater than 0 when it appears so the T you find should be greater than 0, i.e., the negative tail-end of the noise density does not cause a false alarm.

   Solution. Type I error occurs when H_0 is true but we reject it (false alarm). This happens when X is greater than the threshold T under H_0. We solve the equation:

$$\int_T^{\infty} \frac{3}{2}e^{-3|x|}dx = 10^{-5}$$

Since S is greater than 0, we can disregard the negative tail and solve for a positive T:

$$\int_T^\infty \frac{3}{2} e^{-3x} dx = 10^{-5}$$

to get $T = 3.6066$.

b. Using the T you found find the probability of type II error.

Solution. Type II error occurs when H_0 is false but we failed to reject it (missed detection). This happens when X is smaller than the threshold T under H_1. P(type II error) is given by

$$\text{P(type II error)} = \int_{-\infty}^{T-4.5} \frac{3}{2} e^{-3|x|} dx$$

Since $T - 4.5 < 0$, this simplfies to

$$\begin{aligned} \text{P(type II error)} &= \int_{-\infty}^{-0.8934} \frac{3}{2} e^{3x} dx \\ &= 0.0343. \end{aligned}$$

c. What is the power of this test?
Solution. Power $= 1-$ P(type II error) $= 0.9657$.

Problem 3. Let $R_i,\ i = 1, \ldots, 32$ be 32 independent random variables resulting from the envelope detection of a signal plus noise process. Assume that each R_i resulted from the envelope detection of a complex signal plus noise component.

Assuming integration detection with $N = 32$ use Albersheim's equation to plot probability of detection (P_d) vs. SNR (dB) for a probability of false alarm (P_{fa}) of 10^{-6}.

Solution.
We need to plot SNR vs. P_d using Albersheim's equation:

$$SNR = -5 \log_{10} N + (6.2 + \frac{4.54}{\sqrt{N + 0.44}}) \cdot \log_{10}(A + 0.12AB + 1.7B)$$

where $A = \log(\frac{0.62}{P_{fa}})$, $B = \log(\frac{P_d}{1-P_d})$. This can be done with MATLAB:

```
clear all; N=32; Pfa=1e-6;
Pd=0.1:0.05:0.95;
A=log(0.62/Pfa); B=log(Pd./(1-Pd));
SNR=-5*log10(N)+(6.2+4.54/sqrt(N+0.44))*log10(A+0.12*A.*B+1.7*B);
plot(SNR,Pd)
xlabel('SNR(dB)');ylabel('P_d')
```

Figure 2 shows this plot for P_d from 0.1 to 0.95. Note: for $P_d = 0.95$, $SNR = 2.009738$.

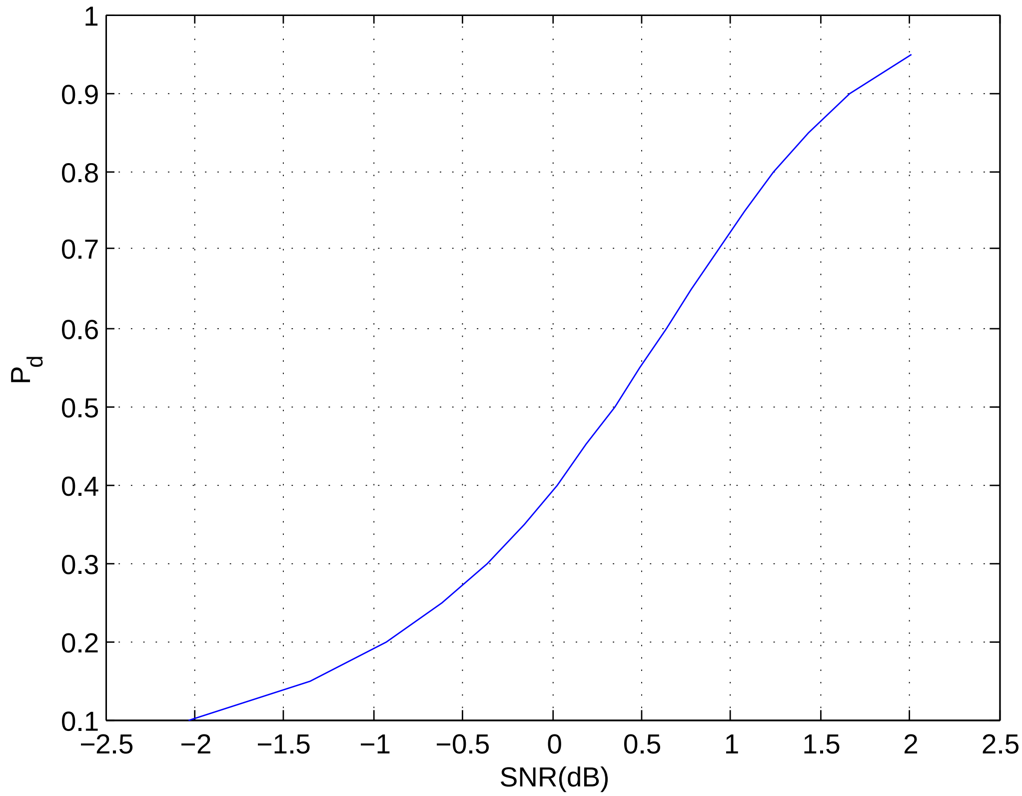

Figure 2: P_d vs. SNR for $N = 32$ and $P_{fa} = 10^{-6}$

Problem 4. Same setup as Problem 3. We know that the density of each R_i when no signal is present is

$$f_{R_i}(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0$$

where σ^2 is the total noise power.

With M of N detection a threshold is found for each of the 32 samples as

$$T_0 = \sqrt{-2\sigma^2 \ln(P_{fa,s})}$$

where $P_{fa,s}$ is the probability of false alarm on a sample basis that yields an overall P_{fa}.

a. Using

$$P_{fa} = \sum_{K=M}^{N} \binom{N}{K} P_{fa,s}^K (1 - P_{fa,s})^{N-K}$$

find $P_{fa,s}$ that yields an overall $P_{fa} = 10^{-6}$ where $M = 16$ and $N = 32$.

b. Using

$$P_d = \sum_{K=M}^{N} \binom{N}{K} P_{d,s}^K (1 - P_{d,s})^{N-K}$$

find $P_{d,s}$, the probability of detection on a sample basis, that yields an overall $P_d = 0.95$ where $M = 16$ and $N = 32$.

c. Using

$$P_{d,s} = \int_{T_0}^{\infty} \frac{r}{\sigma^2} e^{-(r^2+s^2)/2\sigma^2} I_0\left(\frac{rs}{\sigma^2}\right) dr$$

find the $SNR = \frac{s^2}{2\sigma^2}$ required to yield the desired $P_{d,s}$ (express your SNR in dB).

d. Compare the SNR required for a $P_d = 0.95$ in this problem to the SNR required in Problem 3 using integration detection.

Solution. In parts (a) and (b), we need to invert the equations (with $N = 32$ and $M = 16$):

$$\begin{aligned} P_{fa} &= \sum_{K=M}^{N} \binom{N}{K} P_{fa,s}^K (1 - P_{fa,s})^{N-K} \\ P_d &= \sum_{K=M}^{N} \binom{N}{K} P_{d,s}^K (1 - P_{d,s})^{N-K} \end{aligned}$$

in order to determine the $P_{fa,s}$ such that $P_{fa} = 10^{-6}$ and $P_{d,s}$ such that $P_d = 0.95$. Such inversion is impossible analytically. We can instead find the roots of the equations:

$$\begin{aligned} f_1(x) &= 10^{-6} - \sum_{K=M}^{N} \binom{N}{K} x^K (1-x)^{N-K} \\ f_2(x) &= 0.95 - \sum_{K=M}^{N} \binom{N}{K} x^K (1-x)^{N-K} \end{aligned}$$

which can be done numerically. (For example, you can use MATLAB's fzero(.) function to do this). The required $P_{fa,s}$ and $P_{d,s}$ are:

$$\begin{aligned} P_{fa,s} &= 0.1367 \\ P_{d,s} &= 0.6266. \end{aligned}$$

In part (c), we need to invert the equation:

$$P_{d,s} = \int_{T_0}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2+s^2}{2\sigma^2}} I_0\left(\frac{rs}{\sigma^2}\right) dr$$

to find the required $SNR = \frac{s^2}{2\sigma^2}$ to give the $P_{d,s}$ found in part (b). It is easiest to assume $\sigma = 1$ and solve only for s. In this case, $T_0 = \sqrt{-2\sigma^2 \log(P_{fa,s})} = 1.9948$. Solving this equation by the same technique as parts (a) and (b) (and by numerical integration) gives the required SNR to achieve $P_{d,s} = 0.6266$ being:

$$SNR = 3.2634.$$

Note that this is 1.25dB larger than the SNR required by integration method.

Problem 5. Same setup as Problems 3 and 4. In the above problems we used $M = N/2$. Using $P_{fa} = 10^{-6}$ and $P_d = 0.95$ plot required SNR (dB) vs. M for $N = 32$. Use $M = 1, 2, \ldots, 32$ in your plot.
Solution. This can be done with MATLAB.

```
%--------------------------------------------------------------
clear all; SNR=zeros(1,32);
for M=1:32
    Pfas=fzero(@(x) f1(x,M),[0 1]);
```

```
    Pds=fzero(@(x) f2(x,M),[0 1]);
    T=sqrt(-2*log(Pfas));
    S=fzero(@(s) f3(s,T,Pds),5);
    SNR(M)=10*log10(S^2/2);
end
plot(1:M,SNR,'+-'); xlabel('M');ylabel('SNR(dB)')
%--------------------------------------------------------------------
function z = f1(x,M)
N=32;
for K=M:N
    Y(K-M+1)=nchoosek(N,K)*(x^K)*((1-x)^(N-K));
end
z=10^(-6)-sum(Y);
%--------------------------------------------------------------------
function z = f2(x,M)
N=32;
for K=M:N
    Y(K-M+1)=nchoosek(N,K)*(x^K)*((1-x)^(N-K));
end
z=0.95-sum(Y);
%--------------------------------------------------------------------
function z = f3(s,T,Pds)
z=Pds-marcumq(s,T);
%--------------------------------------------------------------------
```

Figure 3 shows the plot of the required SNR vs. M for varying M. We see that the lowest required SNR occurs when $M = 15$, with corresponding required $SNR = 3.2502$.

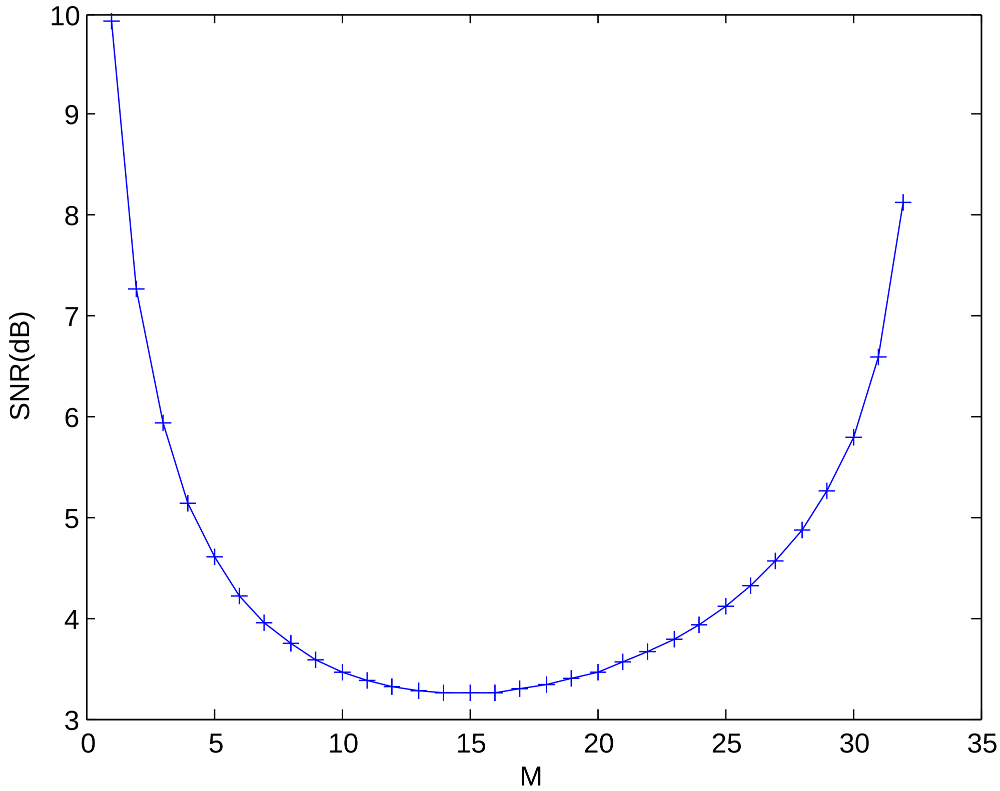

Figure 3: SNR vs. M for N=32, $P_{fa} = 10^{-6}$, $P_d = 0.95$ (M for N detection)