EE 562a

Homework 3

Due Wednesday June 10, 2009

Work the following 10 problems.

Problem 1. Stark and Woods 6.22.

Solution.

$$
\begin{aligned}
X[0] &= 0 \\
X[n] &= \rho X[n-1] + W[n] \;\; for\; n \geq 1. \qquad (1)
\end{aligned}
$$

(a) From (1),

$$
\begin{aligned}
E[X[n]] &= E\left[\sum_{i=1}^{n} \rho^{n-i} W[i]\right] \\
&= \sum_{i=1}^{n} \rho^{n-i} E[W[i]] \\
&= \sum_{i=1}^{n} \rho^{n-i} \cdot 0 \\
&= 0.
\end{aligned}
$$

(b)

$$
\begin{aligned}
K_X[m,n] &= E[(X[m]-\mu)(X[n]-\mu)] \\
&= E[X[m]X[n]] \\
&= E\left[\left(\sum_{i=1}^{m} \rho^{m-i} W[i]\right)\left(\sum_{j=1}^{n} \rho^{n-j} W[j]\right)\right].
\end{aligned}
$$

Since $W[i]$ are independent and mean 0, $E[W[i]W[j]] = 0$, $\forall i \neq j$ and $E[W^2[i]] = \sigma_W^2$.

Let $m \leq n$.

$$
K_X[m,n] = \sum_{i=1}^{m} \rho^{m-i} \rho^{n-i} E[W^2[i]]
$$

$$
\begin{aligned}
&= \sigma_W^2 \rho^{n-m} \sum_{i=1}^{m} (\rho^2)^{m-i} \\
&= \sigma_W^2 \rho^{n-m} \left(\frac{1-\rho^{2m}}{1-\rho^2} \right) \\
&= \sigma_W^2 \frac{\rho^{n-m} - \rho^{m+n}}{1-\rho^2}.
\end{aligned}
$$

In general, for $m, n \geq 1$,

$$
K_X[m,n] = \sigma_W^2 \frac{\rho^{|n-m|} - \rho^{m+n}}{1-\rho^2}.
$$

(c) For $\rho < 1$, $\lim_{m,n\to\infty} \rho^{m+n} = 0$, therefore

$$
\begin{aligned}
\lim_{m,n\to\infty} K_X[m,n] &= \sigma_W^2 \frac{\rho^{|n-m|}}{1-\rho^2} \\
&= G[m-n].
\end{aligned}
$$

Problem 2. Stark and Woods 6.40.

Solution.

$$
\begin{aligned}
X[n] &= X[n-1] + W[n] \\
&= X[n-2] + W[n-1] + W[n] \\
&= \dots \\
&= 0 + W[1] + W[2] + \dots + W[n] \\
&= \sum_{k=1}^{n} W[k].
\end{aligned}
$$

Which is a sum of independent random variables.
(a)

$$
\begin{aligned}
\mu_X[n] &= E\left[\sum_{k=1}^{n} W[k] \right] \\
&= \sum_{k=1}^{n} E[W[k]]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \eta \\
&= n\eta.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_X^2[n] &= Var\left(\sum_{k=1}^{n} W[k]\right) \\
&= \sum_{k=1}^{n} Var(W[k]) \\
&= \sum_{k=1}^{n} \sigma^2 \\
&= n\sigma^2.
\end{aligned}
$$

because $\sum_{k=1}^{n} W[k]$ is a sum of uncorrelated random variables.

(b) Use Chebyshev inequality on $X[n]/n$, with

$$
\begin{aligned}
E\left[\frac{X[n]}{n}\right] &= \frac{1}{n} \cdot n\eta = \eta, \\
Var\left[\frac{X[n]}{n}\right] &= \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}.
\end{aligned}
$$

We have,

$$
P\left[\left|\frac{X[n]}{n} - \eta\right| > \epsilon\right] \leq \frac{\sigma^2}{n\epsilon}
$$

for any fixed $\epsilon > 0$. Thus, as $n \to \infty$, $\lim_{n\to\infty} P\left[\left|\frac{X[n]}{n} - \eta\right| > \epsilon\right] = 0$. Therefore,

$$
\frac{X[n]}{n} \to \eta \ (in\ Prob.).
$$

Problem 3. Let $Z(n)$ be an i.i.d. Bernoulli sequence where

$$
P(Z(n) = 1) = p, \quad P(Z(n) = -1) = q = 1 - p.
$$

Let

$$X(n) = \sum_{k=0}^{n} Z(k)$$

where we take $Z(0) = 0 = X(0)$. Then $X(n)$ is a discrete random walk.

Find $\mathbf{R_X}(n, m)$ for this random walk.

Solution. Assume $m > n$,

$$\begin{aligned}
R_X(m,n) &= E[X(m)X(n)] \\
&= E\left[\sum_{k_1=0}^{n} Z(k_1) \sum_{k_2=0}^{m} Z(k_2)\right] \\
&= \sum_{k=0}^{n} E[Z^2(k)] + \sum_{k_1=0}^{n} \sum_{k_2=0, k_2 \neq k_1}^{m} E[Z(k_1)Z(k_2)] \\
&= n \cdot 1 + nm \cdot (2p-1)^2 \\
&= n[1 + m(2p-1)^2].
\end{aligned}$$

Note:

$$\begin{aligned}
E[Z^2(k)] &= (p \cdot (1)^2 + q \cdot (-1)^2) = p + q = p + (1-p) = 1 \\
E[Z(k_1)Z(k_2)] &= E[Z(k_1)]E[Z(k_2)] = (p \cdot (1) + q \cdot (-1))^2 = (p-q)^2 = (2p-1)^2 \quad for \quad k_1 \neq k_2
\end{aligned}$$

Problem 4. Suppose the discrete random variable U takes on values

$$u = 0, \;\; u = \frac{1}{2}, \;\; u = 1$$

each with probability 1/3. Determine whether each of the following sequences of random variables converge i) surely, ii) almost surely, iii) in mean square or iv) none of these. If the sequence converges in one of these types specify the limiting value.

a. $X_n(u) = \dfrac{u}{n}$.
$\lim_{n\to\infty} X_n(u) = 0 \;\forall\; u$. Therefore $X_n(u)$ converges surely (hence a.s.) to 0.

$$E[|X_n(u) - 0|^2] = E\left[\frac{u^2}{n^2}\right] \to 0.$$

Therefore $X_n(u)$ also converges to 0 in the M.S. sense.

b. $Y_n(u) = u^n$.
$Y_n(u) \to 0$ for $u = 0, \frac{1}{2}$, but $Y_n(u) \to 1$ for $u = 1$. Hence, $Y_n(u)$ does not converge a.s./s. Let

$$Y = \begin{cases} 1 & \text{if } u = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \lim_{n\to\infty} E[|Y_n - Y|^2] &= \lim_{n\to\infty} \left\{E[Y_n^2] - 2E[Y_n Y] + E[Y^2]\right\} \\ &= \frac{1}{3} - 2\left(\frac{1}{3}\right) + \frac{1}{3} = 0. \end{aligned}$$

Hence, Y_n converges to Y in the M.S. sense.

c. $Z_n(u) = cos(4\pi nu)$.
$Z_n(u) = 1 \;\forall\; u, n$. Hence $Z_n(u)$ converges s./a.s. to 1.
$E[|cos(4\pi nu) - 1|^2] = E[0] = 0 \;\forall u, n$. Hence $Z_n(u)$ converges to 1 in the M.S. sense.

d. $W_n(u) = e^{-n(nu-1)}$.
When $u = 0$, $W_n(u) \to \infty$. Hence, $W_n(u)$ does not converge.

Problem 5. Suppose u is selected uniformly in the interval [0, 1]. For each of the following state whether the sequence of random variables converges surely, almost surely or not at all. If the sequence does converge indicate the random variable or constant to which the sequence converges.

a. $\mathbf{X}_n(u) = \dfrac{u}{n}$.
As $n \to \infty$, $X_n(u) \to 0 \;\forall\; u$. Hence, $X_n(u)$ converges a.s. to 0.
$\lim_{n\to\infty} E[|X_n(u) - 0|^2] = \lim_{n\to\infty} E\left[\frac{u^2}{n^2}\right] = 0$. Hence $X_n(u)$ converges to 0 in M.S. sense.

b. $\mathbf{Y}_n(u) = u \cdot \left(1 - \dfrac{1}{n}\right)$.
As $n \to \infty$, $Y_n(u) \to u \;\forall\; u$. Hence, $X_n(u)$ converges a.s. to u.
$\lim_{n\to\infty} E[|Y_n(u) - u|^2] = \lim_{n\to\infty} E\left[\frac{u^2}{n^2}\right] = 0$. Hence $Y_n(u)$ converges to u in M.S. sense.

c. $\mathbf{Z}_n(u) = ue^n$.
As $n \to \infty$, $Z_n(u) \to \infty \; \forall \; u$. Hence, $Z_n(u)$ does not converge.

d. $\mathbf{V}_n(u) = u^n$.
As $n \to \infty$, $V_n(u) \to 0 \; \forall \; u$ except $u = 1$. But $P(u = 1) = 0$. Hence $V_n(u)$ converges a.s. to 0.
$\lim_{n\to\infty} E[|V_n(u) - 0|^2] = \lim_{n\to\infty} E[u^{2n}] = 0$. Hence $V_n(u)$ converges to 0 in M.S. sense.

e. $\mathbf{W}_n(u) = u + u^2 + u^3 + \cdots + u^n$.
$\lim_{n\to\infty} W_n(u) = \frac{1}{1-u} \; \forall \; u$ except $u = 1$. But $P(u = 1) = 0$. Hence $W_n(u)$ converges a.s. to $\frac{1}{1-u}$.

Problem 6. Suppose $X \sim N(0, 6)$ and $Y \sim N(0, 3)$ and X and Y are independent. Use Chebyshev's inequality to bound

$$P(|X - Y| > 8).$$

Solution. Define $Z = X - Y$ then

$$\begin{aligned} Z &\sim N(E[X - Y], E[(X - Y)^2]) \\ &= N(0, E[X^2] + E[Y^2]) \\ &= N(0, 9). \end{aligned}$$

$$\begin{aligned} P(|X - Y| > 8) = P(|Z| > 8) = P(|Z - \mu_Z| > 8) &\leq \frac{\sigma_Z^2}{8^2} \\ &= \frac{9}{64}. \end{aligned}$$

Problem 7. Suppose you flip a fair ($P(head) = P(tail) = 1/2$) nickel coin and let $X_i = 1$ if the ith toss is a head and let $X_i = 0$ if the ith toss is a tail. So $X_1, X_2, \ldots$ are independent Bernoulli random variables each taking the values 0 or 1 with equal probability $p = 1/2$. Let

$$Y_n = X_1 + X_1X_2 + X_1X_2X_3 + X_1X_2X_3X_4 + \cdots + X_1X_2 \cdots X_n.$$

Now suppose you flip a fair dime coin and let Y be the number of tosses required *before* obtaining a heads (so if you get TTH then $Y = 2$ and if you get heads on the first toss then $Y = 0$). Thus

$$P(Y = k) = (1/2)^{k+1}, \quad k = 0, 1, 2, \ldots.$$

a. Determine analytically whether Y_n converges to Y in distribution.
Hint: You might want to work this problem directly instead of trying to show a stronger convergence that implies convergence in distribution.

Solution.
$P(Y_n = k) = P(X_1 = 1, X_2 = 1, \ldots, X_k = 1, X_{k+1} = 0)$. Hence, pmf of Y_n is

$$P(Y_n = k) = \begin{cases} 0 & k < 0 \\ \left(\frac{1}{2}\right)^{k+1} & 0 \le k < n \\ \left(\frac{1}{2}\right)^{k} & k = n \\ 0 & k > n \end{cases}$$

Let $\lfloor z \rfloor$ denote the largest integer that is less than or equal to $z \in \mathbb{R}$. Then,

$$F_Y(z) = \begin{cases} 0 & z < 0 \\ \sum_{k=0}^{\lfloor z \rfloor} \left(\frac{1}{2}\right)^{k+1} & z \ge 0 \end{cases}$$

$$F_{Y_n}(z) = \begin{cases} 0 & z < 0 \\ \sum_{k=0}^{\lfloor z \rfloor} \left(\frac{1}{2}\right)^{k+1} & 0 \le z < n \\ 1 & z \ge n \end{cases}$$

As $n \to \infty$, $F_{Y_n}(z) \to F_Y(z)$. Hence, convergence in distribution.

b. Determine analytically whether Y_n converges to Y in the mean square sense. Note that Y has the same distribution as $X - 1$ where X is a geometric random variable defined as

$$P(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots$$

where $p = 1/2$ and $E(X) = 1/p, \quad Var(X) = (1-p)p^{-2}$.

Solution.
By hint,

$$\begin{aligned} E[Y] &= E[X-1] = 2-1 = 1. \\ E[Y^2] &= E[(X-1)^2] = E[X^2] - 2E[X] + 1 = 6 - 4 + 1 = 3. \end{aligned}$$

From pmf of Y_n, we have

$$\begin{aligned} \lim_{n\to\infty} E[Y_n] &= \lim_{n\to\infty} \sum_{k=0}^{n} kP(Y_n = k) \\ &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} k \left(\frac{1}{2}\right)^{k+1} + \frac{n}{2^n} \right) \\ &= \frac{1}{2} \sum_{k=0}^{n} k \left(\frac{1}{2}\right)^{k} + 0 \\ &= 1. \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} E[Y_n^2] &= \lim_{n\to\infty} \sum_{k=0}^{n} k^2 P(Y_n = k) \\ &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} k^2 \left(\frac{1}{2}\right)^{k+1} + \frac{n}{2^n} \right) \\ &= \frac{1}{2} \sum_{k=0}^{n} k^2 \left(\frac{1}{2}\right)^{k} + 0 \\ &= 3. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{n\to\infty} E[|Y_n - Y|^2] &= \lim_{n\to\infty} \left(E[Y_n^2] - 2E[Y]E[Y_n] + E[Y^2] \right) \\ &= 3 - 2\cdot 1 \cdot 1 + 3 = 4 \neq 0. \end{aligned}$$

Hence, Y_n does not converge to Y in the M.S. sense.

Problem 8. Let X_n be a sequence of i.i.d. equiprobable Bernoulli random variables and let

$$Y_n = 2^n X_1 X_2 \dots X_n.$$

a. Show this sequence converges almost surely and indicate the limit.
$Y_n = 0$ except if $X_1 = X_2 = \ldots = X_n = 1$. Therefore,

$$P(Y_n = k) = \begin{cases} 0 & k = 0 \\ \left(\frac{1}{2}\right)^n & k = 2^n \\ 0 & otherwise \end{cases} \tag{2}$$

From (2), $\lim_{n\to\infty} P(Y_n = 0) = 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$. Therefore, Y_n converge to 0 a.s..

b. Determine whether or not Y_n converges in the mean square sense.
Assume $m > n$,

$$\begin{aligned} E[Y_n^2] &= (2^n)^2 \frac{1}{2^n} = 2^n \\ E[Y_m^2] &= 2^m \\ E[Y_m Y_n] &= (2^n)(2^m)\frac{1}{2^m} = 2^n. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{m,n\to\infty} E[|Y_n - Y_m|^2] &= \lim_{m,n\to\infty} E[Y_n^2] + E[Y_m^2] - 2E[Y_n Y_m] \\ &= \lim_{m,n\to\infty} 2^n + w^m - 2^n \\ &= \lim_{m,n\to\infty} 2^m = \infty. \end{aligned}$$

Hence, Y_n does not converge in the M.S. sense.

Problem 9. A random variable $\mathbf{X}$ is said to be Laplacian with parameter $\alpha > 0$ if it has density

$$f_X(x) = \frac{\alpha}{2}\, e^{-\alpha|x|}, \quad -\infty < x < \infty.$$

Let $\mathbf{X}_n$ be a sequence of Laplacian random variables with parameter $\alpha = n$. Show this sequence converges in probability (and hence in distribution).

Solution.
For all $\epsilon > 0$,

$$\begin{aligned} P(|X_n - 0| \geq \epsilon) &= 1 - \int_{-\epsilon}^{\epsilon} \frac{n}{2} e^{-n|x|} dx \\ &= 1 - n \int_{0}^{\epsilon} e^{-nx} dx \\ &= e^{-n\epsilon} \rightarrow \text{as } n \rightarrow \infty \end{aligned}$$

Therefore X_n converges to 0 in Probability.

Problem 10. Explain why almost sure convergence is the same thing as convergence with probability 1 as defined in class.

Solution.
Convergence with $P = 1$ if $P(|X_n - X| \geq \epsilon_{i.o}) = 0$. Since $|X_n - X| \geq \epsilon_{i.o}$ is the limit superior of the events

$$A_n \triangleq \{|X_n(u) - X(u)| \geq \epsilon\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

where $A_k \triangleq \{u \in \mathcal{U} : |X_n(u) - X(u)| \geq \epsilon\}$.

In other words, the set of events which do not converge has probability 0.

$$\begin{aligned} S^c &= \{u \in \mathcal{U} : X_n(u) \text{ does not converge}\} \\ P(S^c) &= 0, \end{aligned}$$

Hence $P(S) = 1 - 0 = 1$, where $S = \{u \in \mathcal{U} : X_n(u) \text{ converges}\}$, which is convergence a.s..