

EE 562a

Homework 3

Due Wednesday June 10, 2009

Work the following 10 problems.

Problem 1. Stark and Woods 6.22.

Problem 2. Stark and Woods 6.40.

Problem 3. Let $Z(n)$ be an i.i.d. Bernoulli sequence where

$$P(Z(n) = 1) = p, \quad P(Z(n) = -1) = q = 1 - p.$$

Let

$$X(n) = \sum_{k=0}^n Z(k)$$

where we take $Z(0) = 0 = X(0)$. Then $X(n)$ is a discrete random walk.

Find $\mathbf{R}_X(n, m)$ for this random walk.

Problem 4. Suppose the discrete random variable U takes on values

$$u = 0, \quad u = \frac{1}{2}, \quad u = 1$$

each with probability $1/3$. Determine whether each of the following sequences of random variables converge i) surely, ii) almost surely, iii) in mean square or iv) none of these. If the sequence converges in one of these types specify the limiting value.

- a. $X_n(u) = \frac{u}{n}$.
- b. $Y_n(u) = u^n$.
- c. $Z_n(u) = \cos(4\pi nu)$.
- d. $W_n(u) = e^{-n(nu-1)}$.

Problem 5. Suppose u is selected uniformly in the interval $[0, 1]$. For each of the following state whether the sequence of random variables converges surely, almost surely or not at all. If the sequence does converge indicate the random variable or constant to which the sequence converges.

- a. $\mathbf{X}_n(u) = \frac{u}{n}$.
- b. $\mathbf{Y}_n(u) = u \cdot \left(1 - \frac{1}{n}\right)$.
- c. $\mathbf{Z}_n(u) = ue^n$.
- d. $\mathbf{V}_n(u) = u^n$.
- e. $\mathbf{W}_n(u) = u + u^2 + u^3 + \dots + u^n$.

Problem 6. Suppose $X \sim N(0, 6)$ and $Y \sim N(0, 3)$ and X and Y are independent. Use Chebyshev's inequality to bound

$$P(|X - Y|) > 8.$$

Problem 7. Suppose you flip a fair ($P(\text{head}) = P(\text{tail}) = 1/2$) nickel coin and let $X_i = 1$ if the i th toss is a head and let $X_i = 0$ if the i th toss is a tail. So X_1, X_2, \dots are independent Bernoulli random variables each taking the values 0 or 1 with equal probability $p = 1/2$. Let

$$Y_n = X_1 + X_1X_2 + X_1X_2X_3 + X_1X_2X_3X_4 + \dots + X_1X_2 \dots X_n.$$

Now suppose you flip a fair dime coin and let Y be the number of tosses required *before* obtaining a heads (so if you get TTH then $Y = 2$ and if you get heads on the first toss then $Y = 0$). Thus

$$P(Y = k) = (1/2)^{k+1}, \quad k = 0, 1, 2, \dots$$

- a. Determine analytically whether Y_n converges to Y in distribution.
Hint: You might want to work this problem directly instead of trying to show a stronger convergence that implies convergence in distribution.

- b. Determine analytically whether Y_n converges to Y in the mean square sense. Note that Y has the same distribution as $X - 1$ where X is a geometric random variable defined as

$$P(X = k) = (1 - p)^{k-1}p, \quad k = 1, 2, \dots$$

where $p = 1/2$ and $E(X) = 1/p$, $Var(X) = (1 - p)p^{-2}$.

Problem 8. Let X_n be a sequence of i.i.d. equiprobable Bernoulli random variables and let

$$Y_n = 2^n X_1 X_2 \dots X_n.$$

- Show this sequence converges almost surely and indicate the limit.
- Determine whether or not Y_n converges in the mean square sense.

Problem 9. A random variable \mathbf{X} is said to be Laplacian with parameter $\alpha > 0$ if it has density

$$f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}, \quad -\infty < x < \infty.$$

Let \mathbf{X}_n be a sequence of Laplacian random variables with parameter $\alpha = n$. Show this sequence converges in probability (and hence in distribution).

Problem 10. Explain why almost sure convergence is the same thing as convergence with probability 1 as defined in class.