EE 562a

Homework 2 Solutions

Due Wednesday June 3, 2009

Work the following 6 problems.

Problem 1. Given the covariance matrix

$$\mathbf{K} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 12 \\ 3 & 12 & 27 \end{bmatrix}.$$

Factor $\mathbf{K}$ as $\mathbf{K} = \mathbf{H}\mathbf{H}^\dagger$.

Solution.

$$\mathbf{H}\mathbf{H}^\dagger = \begin{bmatrix} h_{11} & 0 & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} & h_{31} \\ 0 & h_{22} & h_{32} \\ 0 & 0 & h_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 12 \\ 3 & 12 & 27 \end{bmatrix}.$$

Solving for each individual element h_{ij}, we have

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}.$$

Problem 2. Let $\mathbf{W}(u)$ be a white random vector with

$$\mu_W = (0 \quad 0 \quad 0)^T, \quad \mathbf{K}_W = \mathbf{I}.$$

Let

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{c}.$$

Find $\mathbf{c}$ and a causal matrix $\mathbf{H}$ using the direct method that produces

$$\mu_X = [4 \ \ 1 \ \ 3]^T, \quad \mathbf{K}_X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution. This is the standard Spectral Shaping problem described in Lecture. We immediately have $\mathbf{c} = \mu_x = [4 \ \ 1 \ \ 3]^T$. And using direct method to solve causal $\mathbf{H}$ such that $\mathbf{K_X} = \mathbf{H}\mathbf{H}^\dagger$. We get the general form of $\mathbf{H}$:

$$H = \begin{bmatrix} a & 0 & 0 \\ a & b & 0 \\ a & b & c \end{bmatrix}.$$

where $a, b, c \in \{+1, -1\}$.

Problem 3. Stark and Woods 5.22. In this problem you can use Matlab or any other software tool you wish.

Solution. In this problem,

$$\mathbf{K_x} = \begin{bmatrix} 2 & -1.5 \\ -1.5 & 2 \end{bmatrix}.$$

$\mathbf{K_x}$ has eigen-decomposition:

$$\begin{aligned} \lambda_1 &= 0.5,\ \mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \\ \lambda_2 &= 3.5,\ \mathbf{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T \end{aligned}$$

The filters used in the simulation and whitening problems are thus:

$$\begin{aligned} \mathbf{H_{sim}} &= \mathbf{E}\mathbf{\Lambda}^{1/2} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{7}}{2} \\ \frac{1}{2} & -\frac{\sqrt{7}}{2} \end{bmatrix}. \\ \mathbf{H_{white}} &= \mathbf{\Lambda}^{-1/2}\mathbf{E}^\dagger = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{7}} & -\frac{1}{\sqrt{7}} \end{bmatrix}. \end{aligned}$$

So we can first generate 1000 realizations of a two dimensional white random vector and produce a scatter diagram using the MATLAB script:

```
W=randn(2,1000); X=zeros(2,1000); Y=zeros(2,1000);
X(1,:)=W(1,:)/2+sqrt(7)*W(2,:)/2;
X(2,:)=W(1,:)/2-sqrt(7)*W(2,:)/2;
scatter(X(1,:),X(2,:),'+');
figure(2)
Y(1,:)=X(1,:)+X(2,:);
Y(2,:)=X(1,:)/sqrt(7)-X(2,:)/sqrt(7);
scatter(Y(1,:),Y(2,:),'+');
```

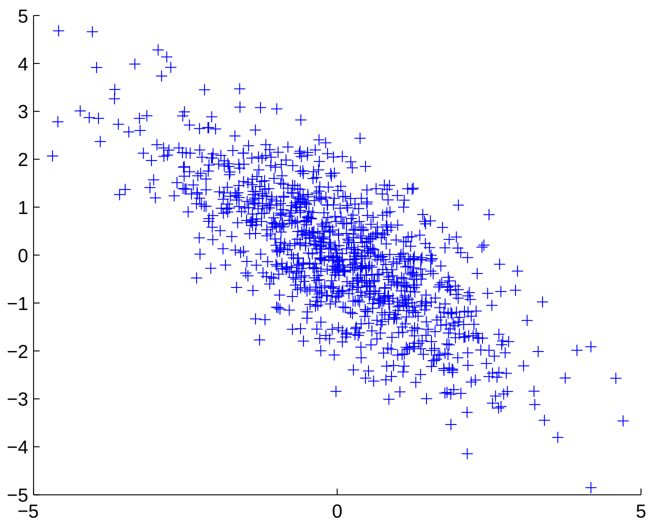

Figure 1: Scatter plot of 1000 2x1 colored random vector

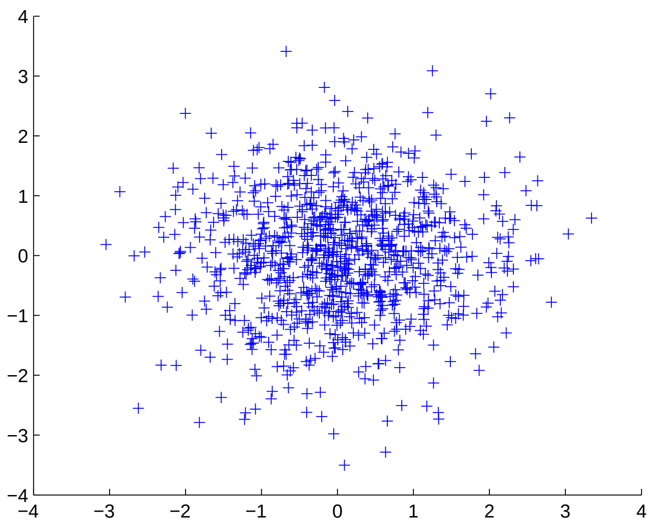

Figure 2: Scatter plot of 1000 2x1 white random vector

Problem 4. Let $\mathbf{X}(u)$ be a random vector with correlation matrix $\mathbf{R_X}$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively. Assume that

$$||\mathbf{e}_1|| = ||\mathbf{e}_2|| = 1.$$

Let

$$Y_i(u) = \mathbf{e}_i^\dagger \mathbf{X}(u), \quad i = 1, 2.$$

a. Compute $E[|Y_1(u)|^2]$.

Note $Y_1(u)$ is scalar, so:

$$\begin{aligned}
\mathbb{E}[|Y_1(u)|^2] &= \mathbb{E}[Y_1(u)^\dagger Y_1(u)] = \mathbb{E}[Y_1(u)Y_1(u)^\dagger] \\
&= \mathbb{E}[\mathbf{e_1^\dagger X(u)(e_1^\dagger X(u))}^\dagger] \\
&= \mathbf{e_1^\dagger R_x e_1} \\
&= \lambda_1 ||\mathbf{e_1}||^2 = \lambda_1.
\end{aligned}$$

b. Compute $E[Y_1(u)Y_2(u)^*]$.

$$\begin{aligned}
\mathbb{E}[Y_1(u)Y_2(u)^*] &= \mathbb{E}[\mathbf{e_1^\dagger X(u)(e_2^\dagger X(u))}^\dagger] \\
&= \mathbf{e_1^\dagger R_x e_2} = \lambda_2 \mathbf{e_1^\dagger e_2} = 0.
\end{aligned}$$

Problem 5. Suppose $\mathbf{X}$ is a mean-zero random vector with covariance matrix

$$\mathbf{K_X} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 12 \\ 3 & 12 & 27 \end{bmatrix}.$$

Find a transformation $\mathbf{A}$ such that $\mathbf{Y} = \mathbf{AX}$ has covariance matrix $\mathbf{K_Y} = \mathbf{I}$.

Solution. Note that this is the same covariance matrix as that in Problem 1. Hence, from Problem 1, one possible matrix $\mathbf{H}$ that factorizes $\mathbf{K_x} = \mathbf{HH}^\dagger$ is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}.$$

Hence a possible whitening filter is,

$$\mathbf{A} = \mathbf{H}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/3 \end{bmatrix}.$$

Problem 6. Suppose $Z \sim N(0,1)$ and let $Y = XZ$ where X is ± 1 Bernoulli with $P(X=1) = P(X=-1) = 1/2$.

a. Show $Y \sim N(0,1)$ by deriving the density function of Y.

Note that

$$F_{Y|X=1}(y) = P(Y \le y|X=1) = P(Z \le y) = F_Z(y)$$

and, since the distribution of Z is symmetrical about 0,

$$\begin{aligned} F_{Y|X=-1}(y) = P(Y \le y|X=-1) = P(-Z \le y) &= P(Z \ge -y) \\ &= P(Z \le y) \\ &= F_Z(y) \end{aligned}$$

Hence the unconditional density function of Y is,

$$\begin{aligned} F_Y(y) &= F_{Y|X=1}(y)P(X=1) + F_{Y|X=-1}(y)P(X=-1) \\ &= \frac{1}{2}F_Z(y) + \frac{1}{2}F_Z(y) \\ &= F_Z(y) \sim N(0,1). \end{aligned}$$

b. Show $W = Z + Y$ is not normal.

Note that

$$W = Z + Y = Z + XZ = Z(1+X).$$

. Hence, when $X = -1$, $W = 0$ and

$$F_{W|X=-1}(w) = U(w).$$

where $U(w)$ is the unit step function.

Also,

$$F_{W|X=1}(w) = P(W \le w|X=1) = P(2Z \le w) = P(Z \le \frac{w}{2}) = F_Z(\frac{w}{2}).$$

Hence the unconditional density function of W is,

$$\begin{aligned} F_W(w) &= F_{W|X=1}(w)P(X=1) + F_{W|X=-1}(w)P(X=-1) \\ &= \frac{1}{2}F_Z(\frac{w}{2}) + \frac{1}{2}U(w). \end{aligned}$$

which is not the CDF of a normal distribution.