EE 562a

Homework 1 Solutions

Due Wednesday May 27, 2009

Work the following 8 problems.

Problem 1. Suppose the random variable X has density

$$f(x) = \begin{cases} 4e^{-4x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Compute $P(X < 0.5)$.

$$\begin{aligned} P(X < 0.5) &= \int_{-\infty}^{0.5} f(x)dx = \int_0^{0.5} 4e^{-4x}dx \\ &= 1 - e^{-2} \end{aligned}$$

b. Let $Y = X^2 + X + 1$. Find $E(Y)$.

$$\begin{aligned} E[Y] &= E[X^2 + X + 1] = \int_{-\infty}^{\infty} (x^2 + x + 1)f(x)dx \\ &= \int_0^{\infty} (x^2 + x + 1)4e^{-4x}dx \\ &= \int_0^{\infty} x^2 4e^{-4x}dx + \int_0^{\infty} x4e^{-4x}dx + \int_0^{\infty} 4e^{-4x}dx \\ &= \frac{11}{8} \end{aligned}$$

Problem 2. Compute the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

As per Stark & Woods 5.4-1, we find that:

$$\begin{aligned} \lambda_1 &= 1,\ \phi_1 = \frac{1}{\sqrt{6}}[1, 2, 1]^T \\ \lambda_2 &= 3,\ \phi_2 = \frac{1}{\sqrt{2}}[-1, 0, 1]^T \\ \lambda_3 &= 4,\ \phi_3 = \frac{1}{\sqrt{3}}[-1, 1, -1]^T \end{aligned}$$

Problem 3. Suppose $\mathbf{Y}(u) = \mathbf{A}^t\mathbf{X}(u) + b$ where

$$\mathbf{A}^t = [3 \quad -1 \quad 3], \;\; \mu_X^t = [2 \quad -3 \quad 4], \;\; b = 8$$

and let

$$\mathbf{K}_X = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$

a. Compute the mean of $\mathbf{Y}(u)$.
   $\mu_Y = \mathbf{A^t}\mu_X + b = 29.$

b. Compute the variance of $\mathbf{Y}(u)$.
   $\sigma_Y^2 = \mathbf{K_Y} = \mathbf{A^tK_XA} = 56.$

Problem 4. Let $\mathbf{Y}(u) = \mathbf{HX}(u)$ where $\mathbf{H}$ is a unitary matrix ($\mathbf{H}^\dagger\mathbf{H} = \mathbf{I}$). Suppose $\mathbf{e}$ is an eigenvector of $\mathbf{K}_X$.

a. Show $\mathbf{He}$ is an eigenvector of $\mathbf{K}_Y$. Assume λ is the eigenvalue associated with $\mathbf{e}$. From the problem, we have $\mathbf{HK_XH}^\dagger$ and $\mathbf{H}^\dagger\mathbf{H} = I$. Hence we have,

$$\mathbf{K_YHe} = \mathbf{HK_XH}^\dagger\mathbf{He} = \mathbf{HK_Xe} = \mathbf{H}\lambda\mathbf{e} = \lambda\mathbf{He}.$$

So $\mathbf{He}$ is an eigenvector of $\mathbf{K_Y}$ with the same eigenvalue λ.

b. Show $||\mathbf{He}|| = ||\mathbf{e}||$.

$$||\mathbf{He}|| = \sqrt{(\mathbf{He})^\dagger(\mathbf{He})} = \sqrt{\mathbf{e}^\dagger\mathbf{H}^\dagger\mathbf{He}} = \sqrt{\mathbf{e}^\dagger\mathbf{e}} = ||\mathbf{e}||.$$

Problem 5. Show

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1.$$

Hint: Let

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Show $I^2 = 1$ by converting to polar coordinates.

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy.$$

Let

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & -r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r. \end{aligned}$$

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r\,dr\,d\theta, \\ &= \int_0^{2\pi} \frac{1}{2\pi} d\theta \cdot \int_0^{\infty} r e^{-\frac{r^2}{2}} dr, \\ &= 1 \cdot \left[-e^{-\frac{r^2}{2}} \Big|_0^{\infty} \right], \\ &= 1 \cdot (0 - (-1)) = 1. \end{aligned}$$

$$\Longrightarrow I = 1.$$

In class we will often need to move limits inside of integrals. The justification for this will be given when needed. In the meantime consider the following two problems:

Problem 6. Give an example of a sequence of functions $f_n(x)$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx.$$

Solution. Let

$$f_n(x) = n\chi_{[0,1/n)}(x),$$

where $\chi_A(x) = 1$ if $x \in A$ and is 0 elsewhere. Note that $f_n(x) \to 0$ pointwise, i.e., for all real numbers x, $f_n(x)$ is eventually 0 for n large enough. Hence,

$$\lim_{n\to\infty} f_n(x) = 0$$

for all real x. Thus,

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx = 0.$$

But,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \lim_{n\to\infty} \int_0^{1/n} ndx = \lim_{n\to\infty} 1 = 1.$$

Problem 7. Give an example of a sequence of functions $f_n(x)$ such that

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)dx \neq \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)dx.$$

Solution. Let

$$f_n(x) = \chi_{[n,n+1)}(x) - \chi_{[n+1,n+2)}(x).$$

Observe that

$$f_1(x) = \chi_{[1,2)}(x) - \chi_{[2,3)}(x)$$
$$f_2(x) = \chi_{[2,3)}(x) - \chi_{[3,4)}(x)$$

etc., so that $\sum_{n=1}^{\infty} f_n(x)$ converges to $\chi_{[1,2)}(x)$ pointwise, i.e., there exists an N such that for all $n \geq N$ $f_n(x) = \chi_{[1,2)}(x)$ for all real numbers x. Hence,

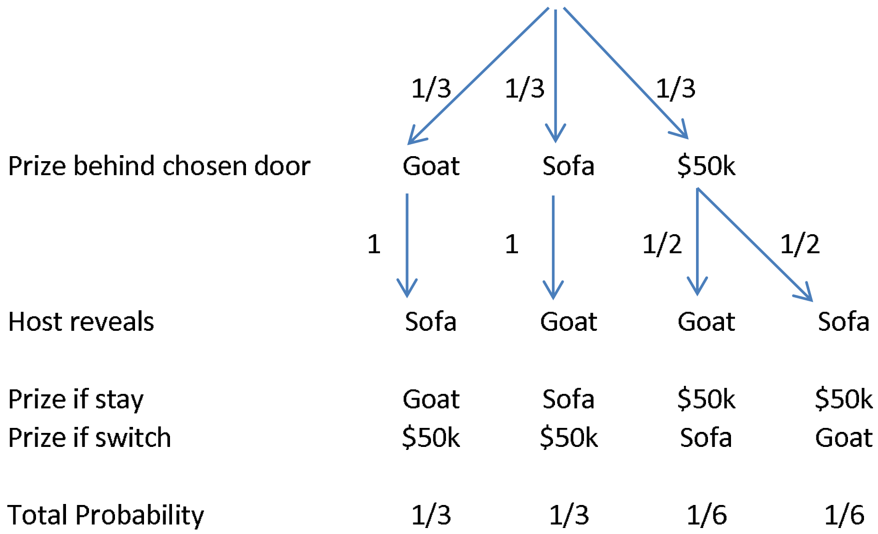
$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} \chi_{[1,2)}(x)dx = \int_1^2 dx = 1.$$

But,

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} [\chi_{[n,n+1)}(x) - \chi_{[n+1,n+2)}(x)]dx = \sum_{n=1}^{\infty} 0 = 0.$$

Problem 8. Alice is a contestant on a popular game show. The host has three doors from which Alice gets to choose. Behind one door is the grand prize (a new car worth $50,000), behind another door is a sofa and behind another door is a goat. The host knows what is behind each door but Alice cannot see what is behind any of the doors. Alice does know though that one of the doors has a grand prize, one has a fairly nice prize and one has a joke prize. Alice picks door number 1. Before the host reveals to Alice and the audience what is behind door number 1, he opens door number 2 and reveals a sofa prize (the host always opens a door that does not have the grand prize and can do this because he knows what is behind each door). He then asks Alice if she wants to exchange her door with door number 3. Her friend Bob sitting in the audience urges her to keep her door. Her friend Carol urges her to exchange her door for door number 3. Her friend Dave just sits quietly because he thinks it does not matter if she keeps her door or exhanges it. Who do you think is right (Bob, Carol or Dave)? That is, to maximize her probabilty of winning the grand prize should Alice keep her door, exchange it or does it matter? First try to answer this question using your intuition then justify your answer mathematically using some conditional probability calculations.

Solution.



P(win \$50k by staying | Host reveals door without prize) $= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.
P(win \$50k by switching | Host reveals door without prize) $= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

Hence, Carol is right, i.e. Alice should switch.