Provided Data. Some of the following may be useful on this test.

Laplace Transforms:

$$1 \longleftrightarrow \frac{1}{s}, \qquad e^{-at} \longleftrightarrow \frac{1}{s+a}, \qquad \delta(t) \longleftrightarrow 1, \qquad t \longleftrightarrow \frac{1}{s^2},$$

if $y(t) \longleftrightarrow Y(s)$ then $y\prime(t) \longleftrightarrow sY(s) - Y(0)$.

Problem 1. We are given an observation of X and we must decide between two hypotheses:

$$H_0 : \quad X = N$$
$$H_1 : \quad X = S_1 + N$$

where, N has density

$$f_N(x) = \begin{cases} 3x^2, \ 0 \le x \le 1 \\ 0, \qquad\qquad\qquad \text{elsewhere} \end{cases}$$

and $S_1 = 0.97$.

a. Find a threshold T (to 3 decimal places) so that the probability of type I error is 0.05.

Solution.

$$\int_T^1 3x^2 dx = 0.05 \Rightarrow T = 0.983.$$

b. Using $T = .952$ (which is not the answer to part (a)) find the probability of type II error.

Solution. With $S = 0.97$ the received signal is always to the right of the threshold of 0.952 so the probability of type II error is 0.

Problem 2. Let $Z(n)$ be a random sequence where

$$P(Z(n) = 1) = 1/2, \quad P(Z(n) = -1) = 1/2$$

and

$$E[Z(n)Z(m)] = -1/2, \ n \ne m.$$

For $n \ge 1$ let

$$X(n) = \sum_{k=0}^{n-1} Z(k)$$

where we take $Z(0) = 0 = X(0)$. Find $\mathbf{R_X}(n, m)$.

Solution. First note that

$$E[Z(n) = 0, \quad E[Z^2(n)] = 1.$$

Now

$$
\begin{aligned}
\mathbf{R}_{\mathbf{X}}(n,m) &= E[X(n)X(m)] = E\left[\sum_{k=0}^{n-1} Z(k) \sum_{l=0}^{n-1} Z(l)\right] \\
&= \sum_{k=1}^{n-1} E[Z(k)] \sum_{l=1}^{m-1} E[Z(l)] = \sum_{k=1}^{\min(n,m)-1} E[Z^2(k)] + \sum_{\substack{k=1 \\ k \neq l}}^{n-1} \sum_{l=1}^{m-1} E[Z(k)Z(l)] \\
&= [\min(n,m) - 1] + [(m-1)(n-1) - \min(n,m) + 1](-1/2) \\
&= \frac{3}{2}[\min(n,m) - 1] - \frac{1}{2}(n-1)(m-1).
\end{aligned}
$$

Problem 3. Two random variables X and Y have joint density

$$f_{XY}(x,y) = xe^{-x(y+1)}u(x)u(y)$$

where $u(x)$ and $u(y)$ are unit step functions. Find the best (possibly nonlinear) estimate in the mean square sense for Y given X.

Solution.

$$E[Y|X=x] = \int_{-\infty}^{\infty} f_{Y|X}(y|x)dy.$$

Now

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

where,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} xe^{-x(y+1)}u(x)u(y)dy = e^{-x}u(x).$$

Thus,

$$f_{Y|X}(y|x) = xe^{-x(y+1)}u(x)u(y)e^{-x}u(x) = xe^{-xy}u(x)u(y).$$

Hence,

$$E[Y|X=x] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x) = \int_{0}^{\infty} yxe^{-xy}dy = \frac{1}{x} \Rightarrow \hat{Y} = \frac{1}{X}.$$

Problem 4. Consider a real Gaussian random sequence $x(n)$, n an integer, with

$$E[x(n)] = 0, \quad E\left[x(n)^2\right] = 1, \quad E[x(n)x(m)] = \rho^{|n-m|}$$

where $0 < \rho < 1$. Let

$$y(n) = x(n) + 1.$$

a. Is $x(n)$ wide sense stationary?

   Solution. Yes.

b. Find the covariance of $y(n)$ and state whether or not it is wide sense stationary.

   Solution. $K_Y(n,m) = \rho^{|n-m|}$. It is wide sense stationary.

c. Does $x(n)$ converge in the mean square sense and, if so, what is the limit?

   Solution. Using Cauchy we find

$$\begin{aligned} E[|x(n) - x(m)|^2] &= E[x^2(n)] + E[x^2(m)] + 2E[x(n)x(m)] \\ &= 1 + 1 + 2\rho^{|n-m|} \not\to 0 \text{ as } n \to \infty \end{aligned}$$

   so $x(n)$ does not converge in the mean square sense.

d. Does $y(n)$ converge in the mean square sense and, if so, what is the limit?

   Solution. Similar argument as in c shows $y(n)$ does not converge in the mean square sense.

e. Does $x(n)$ converge in distribution and, if so, what is the limit?

   Solution. Yes. $x(n) \sim N(0,1)$.

f. Does $y(n)$ converge in distribution and, if so, what is the limit?

   Solution. Yes. $y(n) \sim N(1,1)$.

Problem 5. Let $X(t)$ be a random process with mean μ_X and covariance function

$$K_X(t_1, t_2) = \sigma^2 \cos \omega_0 (t_1 - t_2).$$

a. Show that the mean-square derivative $X'(t)$ exists.

Solution. Let $\tau = t_1 - t_2$. Then

$$K_X(\tau) = \sigma^2 \cos \omega_0(\tau).$$

$$\frac{dK_X(\tau)}{d\tau} = -\sigma^2 \omega_0 (\sin \omega_0(\tau))$$

$$\frac{d^2 K_X(\tau)}{d\tau^2} = -\sigma^2 \omega_0^2 (\cos \omega_0(\tau))$$

which exists for all τ so $X'(t)$ exists.

b. Find the covariance function $K_{X'}(t_1, t_2)$.

Solution.

$$K_{X'}(t_1, t_2) = K_{X'}(\tau) = -\frac{d^2 K_X(\tau)}{d\tau^2} = \sigma^2 \omega_0^2 \cos \omega_0(\tau).$$

Problem 6. Consider a wide-sense stationary random process $X(t)$ with autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T}, & |\tau| \le T \\ 0, & |\tau| > T. \end{cases}$$

Find $S_X(f)$, the power spectral density of $X(t)$.

Solution. Let

$$Rect(\tau) = \begin{cases} 1, & |\tau| \le T/2 \\ 0, & |\tau| > T/2. \end{cases}$$

Then

$$\begin{aligned} S_X(f) = F[R_X(\tau)] &= F\left[\frac{1}{\sqrt{T}} Rect(\tau) * \frac{1}{\sqrt{T}} Rect(\tau)\right] \\ &= F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right] \cdot F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right]. \end{aligned}$$

Now

$$F\left[\frac{1}{\sqrt{T}}Rect(\tau)\right] = \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} e^{-i2\pi ft}dt = \frac{1}{\sqrt{T}}\frac{\sin\pi fT}{\pi f}.$$

Hence,

$$S_X(f) = \frac{1}{T}\left(\frac{\sin\pi fT}{\pi f}\right)^2.$$

Problem 7. The wide sense stationary process $X(t) = X(u,t)$ has mean $\mu_X = 2$ and correlation function

$$R_X(\tau) = 4 + e^{-|\tau|}.$$

Let

$$I = \int_1^3 X(t)dt.$$

a. Find $E[I]$.

Solution.

$$E[I] = \int_1^3 \mu_X dt = \int_1^3 2dt = 4.$$

b. Find $Var(I)$.

Solution. Note that $K_X(\tau) = R_X(\tau) - \mu_X^2 = e^{-|\tau|}$. Let $\tau = t_1 - t_2$. Then

$$K_X(t_1,t_2) = e^{-|t_1-t_2|}.$$

Hence,

$$Var(I) = \int_1^3\int_1^3 K_X(t_1,t_2)dt_1dt_2.$$

Let $s = t_1 + t_2$. Then $-2 \le \tau \le 2$ and $-2 \le s \le 2$ and the absolute value of the Jacobian of the transformation is 2. So

$$\begin{aligned} Var(I) &= \frac{1}{2}\int_{-2}^{2}\int_{-(2-|\tau|)}^{2-|\tau|} e^{-|\tau|}dsd\tau \\ &= \int_{-2}^{2}(2-|\tau|)\,e^{-|\tau|}d\tau \\ &= 2\int_0^2(2-\tau)e^{-\tau} \\ &= 2(1+e^{-2}). \end{aligned}$$

Problem 8. Consider the mean square differential equation

$$\frac{dY(t)}{dt} + Y(t) = X(t)$$

for $t > 0$ subject to the initial condition $Y(0) = 0$. The input is

$$X(t) = e^{-2t} + W(t)$$

where $W(t)$ is a white Gaussian noise process with mean zero and covariance function $K_W(\tau) = \sigma^2\delta(\tau)$.

a. Find the mean function $\mu_Y(t)$.

Solution.
Then taking expectations of both sides we get

$$\frac{d\mu_Y(t)}{dt} + \mu_Y(t) = \mu_X(t)$$

with initial condition $\mu_Y(0) = E[Y(0)] = 0$ and

$$\mu_X(t) = E[X(t)] = E\left[e^{-2t} + W(t)\right] = e^{-2t}.$$

Taking Laplace transforms we get

$$s\mu_Y(s) - \mu_Y(0) + \mu_Y(s) = \mu_X(s)$$

or

$$\mu_Y(s)(s+1) = \frac{1}{s+2}$$

so

$$\mu_Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Thus,

$$\mu_Y(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

b. Find the covariance function $K_Y(t_1, t_2)$.

Solution.

Now let us rewrite our original equation using the variable t_1 instead of t:

$$\frac{dY(t_1)}{dt_1} + Y(t_1) = X(t_1), \quad t_1 \geq 0.$$

Since

$$\frac{d\mu_Y(t_1)}{dt_1} + \mu_Y(t_1) = \mu_X(t_1)$$

we can write

$$\frac{d(Y(t_1) - \mu_Y(t_1))}{dt_1} + (Y(t_1) - \mu_Y(t_1)) = (X(t_1) - \mu_X(t_1)), \quad t_1 \geq 0.$$

Multiplying by $(X^*(t_2) - \mu_X^*(t_2))$ we get

$$\frac{d\left(Y(t_1) - \mu_Y(t_1)\right)\left(X^*(t_2) - \mu_X^*(t_2)\right)}{dt_1} + (Y(t_1) - \mu_Y(t_1))\left(X^*(t_2) - \mu_X^*(t_2)\right)$$

$$= (X(t_1) - \mu_X(t_1))\left(X^*(t_2) - \mu_X^*(t_2)\right).$$

Taking expectations this becomes

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + K_{YX}(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)$$

with initial condition $K_{YX}(0, t_2) = 0$.

For $t_1 < t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + K_{YX}(t_1, t_2) = 0$$

with initial condition $K_{YX}(0, t_2) = 0$ which implies $K_{YX}(t_1, t_2) = 0$.

For $t_1 \geq t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + K_{YX}(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)$$

due to the jump of σ^2 at $t_1 = t_2$. Taking Laplace transforms we get

$$s_1 K_{YX}(s_1, t_2) + K_{YX}(s_1, t_2) = \sigma^2 e^{-s_1 t_2}$$

so

$$K_{YX}(s_1, t_2) = \frac{\sigma^2}{s_1 + 1} e^{-s_1 t_2}.$$

Hence

$$K_{YX}(t_1, t_2) = \sigma^2 e^{-(t_1 - t_2)}$$

for $t_1 \geq t_2$ and is zero otherwise.

We will now derive this last result another way. Instead of considering $t_1 \geq t_2$ we will consider $t_1 > t_2$ with initial condition at $t_1 = t_2$.

For $t_1 > t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + K_{YX}(t_1, t_2) = 0$$

with initial condition $K_{YX}(t_2, t_2) = \sigma^2$.

Taking Laplace transforms we get

$$s_1 K_{YX}(s_1, t_2) - K_{YX}(t_2, t_2) e^{-s_1 t_2} + K_{YX}(s_1, t_2) = 0$$

or

$$s_1 K_{YX}(s_1, t_2) - \sigma^2 e^{-s_1 t_2} + K_{YX}(s_1, t_2) = 0$$

so

$$K_{YX}(s_1, t_2) = \frac{\sigma^2}{s_1 + 1} e^{-s_1 t_2}.$$

Hence

$$K_{YX}(t_1, t_2) = \sigma^2 e^{-(t_1 - t_2)}$$

for $t_1 \geq t_2$ and is zero otherwise. This is the same answer as before.

Repeating the above procedure but now multiplying by $(Y^*(t_2) - \mu_Y^*(t_2))$ we get,

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + K_Y(t_1, t_2) = K_{XY}(t_1, t_2)$$

with initial condition $K_Y(0, t_2) = 0$, or

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + K_Y(t_1, t_2) = K_{YX}(t_2, t_1)$$
$$= \sigma^2 e^{-(t_2 - t_1)}$$

for $t_2 \geq t_1$ and is zero otherwise.

So for $0 < t_1 \leq t_2$

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + K_Y(t_1, t_2) = \sigma^2 e^{-(t_2 - t_1)}.$$

Taking Laplace transforms we get

$$s_1 K_Y(s_1, t_2) + K_Y(s_1, t_2) = \frac{\sigma^2 e^{-t_2}}{s_1 - 1}$$

or

$$K_Y(s_1, t_2) = \frac{\sigma^2 e^{-t_2}}{(s_1 - 1)(s_1 + 1)} = \frac{\sigma^2 e^{-t_2}/2}{s_1 - 1} - \frac{\sigma^2 e^{-t_2}/2}{s_1 + 1}$$

so

$$K_Y(t_1, t_2) = \frac{\sigma^2}{2} e^{-t_2} \left(e^{t_1} - e^{-t_1} \right).$$

For $t_1 > t_2$

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + K_Y(t_1, t_2) = 0$$

with initial condition $K_Y(t_2, t_2) = \frac{\sigma^2}{2} \left(1 - e^{-2t_2}\right).$

Taking Laplace transforms we get

$$s_1 K_Y(s_1, t_2) - K_Y(t_2, t_2) e^{-s_1 t_2} + K_Y(s_1, t_2) = 0$$

or

$$s_1 K_Y(s_1, t_2) - \frac{\sigma^2}{2} \left(1 - e^{-2t_2}\right) e^{-s_1 t_2} + K_Y(s_1, t_2) = 0$$

so

$$K_Y(s_1, t_2) = \frac{\sigma^2}{2} \frac{(1 - e^{-2t_2})}{s_1 + 1} e^{-s_1 t_2}$$

Taking the inverse tranform we get

$$K_Y(t_1, t_2) = \frac{\sigma^2}{2} \left(1 - e^{-2t_2}\right) e^{-(t_1 - t_2)}.$$

Problem 9. Show how to use the inverse transform method to generate a random variable X having density function

$$f(x) = \begin{cases} 1 - \dfrac{x}{2}, & 0 \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Solution.

$$F_X(x) = \int_0^x \left(1 - \frac{t}{2}\right) dt = \begin{cases} 0, & x < 0 \\ x - \dfrac{x^2}{4}, & 0 \le x \le 2 \\ 1, & x > 2. \end{cases}$$

Set

$$u = x - \frac{x^2}{4}$$

to get

$$x = 2(1 - \sqrt{1-u})$$

or since $1 - u$ is uniform we obtain

$$x = 2(1 - \sqrt{u}).$$