EE 503

Lecture Notes Part 15

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15.0 Wiener, Poisson and Markov Processes and Markov C

15.1 The Wiener Process

Let $z(n)$ be a Bernoulli sequence where

$$P(z(n) = s) = P(z(n) = -s) = 1/2$$

where s is the step size. We have

$$E[z(n)] = 0, \quad E[z^2(n)] = s^2.$$

Let

$$x(n) = \sum_{k=1}^{n} z(k).$$

Then, $x(n)$ is a discrete random walk. Now

$$P\left(x(n) = rs\right) = P(k \text{ successes and } (n-k) \text{ failures in } n \text{ trials})$$

where,

$$rs = ks - (n-k)s$$

i.e.,

$$k = \frac{r+n}{2}, \quad \text{an integer.}$$

So

$$P\left(x(n) = rs\right) = P\left(\frac{r+n}{2} \text{ successes in } n \text{ trials}\right)$$

$$= \binom{n}{\frac{n+r}{2}} 2^{-n}, \quad \frac{n+r}{2} \text{ an integer.}$$

Now

$$E[x(n)] = E\left[\sum_{k=1}^{n} z(k)\right] = 0$$

$$Var[x(n)] = E[x^2(n)] = E\left[\left(\sum_{k=1}^{n} z(k)\right)^2\right] = n \cdot Var[z(k)] = ns^2.$$

Define the RP with index set $T = [0, \infty)$ as follows:

$$X_\tau(t) = \sum_{k=1}^{\infty} z(k) u(t - k\tau)$$

where, τ is a time interval, s is a step size.

The Wiener process is formed by letting $\tau, s \to 0$ so that the limit is a continuous time RP (sample paths are continuous) and the variance is not trivial (i.e., $\neq 0, \neq \infty$).

Now

$$E[X_\tau(t)] = 0, \quad E[X_\tau^2(t)|_{t=n\tau}] = ns^2.$$

So

$$E[X_\tau^2(t)] = s^2 t/\tau.$$

Let $s^2 = \alpha\tau$. Then

$$E[X_\tau^2(t)] = \frac{\alpha\tau t}{\tau}.$$

We get

$$\lim_{\tau \to 0} E[X_\tau^2(t)] = \alpha t.$$

Let

$$X(t) = \lim_{\tau \to 0} X_\tau(t).$$

$X(t)$ is a Wiener process.

By the central limit theorem the first order density is $\sim N(0, \alpha t)$. We get

$$f_X(x; t) = \frac{1}{\sqrt{2\pi\alpha t}} \exp\left(-\frac{x^2}{2\alpha t}\right), \ t > 0.$$

For any $0 \leq t' < t$, the increment

$$\Delta = X(t) - X(t')$$

is a random variable having a Gaussian distribution with mean zero and variance $\alpha(t - t')$. Note

$$E[X(t) - X(t')] = 0, \ E[(X(t) - X(t'))^2] = \alpha(t - t'), \ t > t'.$$

The increment is independent of $X(\hat{t}) \ \forall \ \hat{t} \leq t'$.

Covariance

$$K_X(t,s) = E[X(t)X(s)].$$

For $t > s$ compute

$$E[(X(t) - X(s))X(s)] = E[X(t) - X(s)]E[X(s)] = 0.$$

So

$$E[X(t)X(s)] = E[X^2(s)] = \alpha s,\ t > s$$

and similarly

$$E[X(t)X(s)] = E[X^2(t)] = \alpha t,\ t \leq s$$

Thus

$$K_X(t,s) = \alpha \min(t,s).$$

All nth order pdf's are Gaussian. Hence, the Wiener process is a special Gaussian process. It turns out the mean square derivative of a Wiener process is white noise.

15.2 The Poisson Process

Consider a sequence of i.i.d. random variables $\gamma(n),\ n \geq 1$, with density

$$f_\gamma(t,n) = \lambda e^{-\lambda t} u(t), \quad n = 1, 2, \ldots.$$

Define

$$T(n) = \sum_{k=1}^{n} \gamma(k).$$

$T(n)$ would represent the time of arrival of the nth event if $\gamma(n)$ represents the interarrival times. This is used in modeling counts on a Geiger counter that detects particles and is also used in Queuing theory.

Now $T(n)$ is the sum of n i.i.d. random variables so its pdf is the $(n-1)$ fold convolution of $f_\gamma(t,n)$. We get

$$f_T(t,n) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} u(t)$$

$$E[T(n)] = E\left[\sum_{k=1}^{n} \gamma(k)\right] = n/\lambda$$

$$Var[T(n)] = n/\lambda^2 = nVar[\gamma(n)].$$

Define

$$N(t) = \sum_{n=1}^{\infty} u[t - T(n)]$$

which equals the number of arrivals (or events) up to and including time t. Now

$$\gamma(n) = T(n) - T(n-1)$$

$$P[N(t) = n] = P[T(n) \leq t,\ T(n+1) > t]$$

$$= P[T(n) \leq t,\ \gamma(n+1) > t - T(n)]$$

$$= \int_0^t f_T(\alpha, n) \int_{t-\alpha}^{\infty} f_\gamma(\beta, n+1) d\beta d\alpha$$

$$= \int_0^t \frac{\lambda^n \alpha^{n-1} e^{-\lambda t}}{(n-1)!} \int_{t-\alpha}^{\infty} \lambda e^{-\lambda\beta} d\beta d\alpha \ u(t)$$

$$= \left(\int_0^t \alpha^{n-1} d\alpha\right) \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \ u(t)$$

or

$$P[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \ u(t), \quad t \geq 0.$$

Note that $T(n)$ has independent increments. So

$$P[N(t_b) - N(t_a) = n] = \frac{[\lambda(t_b - t_a)]^n}{n!} e^{-\lambda(t_b - t_a)} \ u(n).$$

Now

$$E[N(t)] = \lambda t.$$

Suppose $t_2 \geq t_1$. Then

$$E[N(t_2)N(t_1)] = E[(N(t_1) + [N(t_2) - N(t_1)])N(t_1)]$$

$$= E[(N(t_1)^2] + E[N(t_2) - N(t_1)]E[N(t_1)]$$

$$\lambda t_1 + \lambda^2 t_1^2 + \lambda(t_2 - t_1)\lambda t_1$$

$$= \lambda t_1 + \lambda^2 t_1 t_2.$$

For $t_1 > t_2$ a similar expression holds. Thus,

$$R_N(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$

and

$$K_N(t_1, t_2) = \lambda \min(t_1, t_2).$$

Example: Radioactivity monitoring that counts particles can often be modeled as Poisson. We start monitoring at time t and count for T_0 seconds.

Let ΔN = number of counts in the interval $[t, t+\tau] = N(t+\tau) - N(t)$. Then ΔN has a Poisson distribution with mean λT_0 where λ is the average arrival rate. The probability that an alarm does not sound is

$$P[\Delta N \le N_0] = \sum_{k=0}^{N_0} \frac{(\lambda T_0)^k}{k!} e^{-\lambda T_0}.$$

15.3 The Markov Process and Markov Chain

A random process $X(t)$ is a Markov Process if the future of the process given the present is independent of the past. Thus, if $X(t)$ is discrete-valued

$$\begin{aligned} P[X(t_{k+1}) &= x_{k+1} | X(t_k) = x_k, \ldots, X(t_1) = x_1] \\ &= P[X(t_{k+1}) = x_{k+1} | X(t_k) = x_k] \end{aligned}$$

and if $X(t)$ is continuous-value

$$\begin{aligned} P[a < X(t_{k+1}) &\le b | X(t_k) = x_k, \ldots, X(t_1) = x_1] \\ &= P[a < X(t_{k+1}) \le b | X(t_k) = x_k]. \end{aligned}$$

Example: The sum process

$$S_n = X_1 + \cdots + X_n = S_{n-1} + X_n$$

where the X_i's are iid and $S_0 = 0$ is a Markov process as shown in class.

Example: The moving average of a (0,1) Bernoulli sequence with $p = 1/2$

$$Y_n = \frac{1}{2}(X_n + X_{n-1})$$

is not a Markov process as demonstrated in class.

Examples: The Poisson process and the Wiener process are each Markov processes.

An integer-valued Markov process is called a Markov chain. If $X(t)$ is a Markov chain then

$$\begin{aligned} P[X(t_{k+1}) &= x_{k+1}, X(t_k) = x_k, \ldots, X(t_1) = x_1] \\ &= \prod_{j=1}^{k} P[X(t_{j+1}) = x_{j+1} | X(t_j) = x_j] P[X(t_1) = x_1]. \end{aligned}$$

Let $X - n$ be a discrete-time Markov chain that starts at $n = 0$ with pmf

$$p_j(0) := P[X_0 = j], \; j = 0, 1, 2, \ldots .$$

We will assume that X_n takes values from a countable set of integers. We say the Markov chain is finite state if X_n takes on values from a finite set. We have

$$P[X_n = i_n, \ldots, X_0 = i_0]$$

$$= P[X_n = i_n | X_{n-1} = i_{n-1}] \cdots P[X_1 = i_1 | X_0 = i_0] P[X_0 = i_0].$$

Then X_n is completely specified by its initial pmf $P_i(0$ and the one-step transition probabilities

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \ldots & \vdots \end{bmatrix}.$$

Example: Consider the two-state Markov chain for speech activity. Here if the nth packet contains silence then the probability of silence in the next packet is $1 - \alpha$ and the probability of speech is α. If the nth packet contains speech then the probability of speech in the next packet is $1 - \beta$ and the probability of silence is β. Thus,

$$P = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}.$$

We will now consider the n-step transition probabilities, $P(n) = \{p_{ij}(n)\}$ where

$$p_{ij}(n) = P[X_{n+k} = j | X_n = i].$$

Note that $P[X_{n+k} = j|X_n = i] = P[X_n = j|X_0 = i]$ since the transition probabilities do not depend on time. We find that

$$P(n) = P^n.$$

Example: in our speech activity example if we let $\alpha = 0.1$ and $\beta = 0.2$ we find

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}, \quad P^{16} = \begin{bmatrix} 0.6678 & 0.3322 \\ 0.6644 & 0.3356 \end{bmatrix}.$$

We can find P^n as $n \to \infty$ by diagonalizing P as $P = E\Lambda E^{-1}$ where E is the matrix of eigenvectors and Λ is the diagonal matrix consisting of eigenvalues. We find

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}, \quad E = \begin{bmatrix} 1 & \alpha \\ 1 & -\beta \end{bmatrix}.$$

Then, $P^n = E\Lambda^n E^{-1}$ which becomes

$$P^n \to \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

as $n \to \infty$.

The example of our speech example is typical of Markov chains in which all the rows of the n-step transition matrix approach the same pmf, that is, $p_{ij}(n) \to \pi_j$ for all i. This means that eventually

$$\pi_j = \sum_i p_{ij}\pi_i$$

which in matrix form becomes

$$\pi = \pi P.$$

In our speech example we get

$$\begin{aligned} \pi_0 &= (1-\alpha)\pi_0 + \beta\pi_1 \\ \pi_1 &= \alpha\pi_0 + (1-\beta)\pi_1 \end{aligned}$$

which means $\alpha\pi_0 = \beta\pi_1 = \beta(1-\pi_0)$ since $\pi_0 + \pi_1 = 1$. Thus,

$$\pi_0 = \frac{\beta}{\alpha+\beta} = \frac{2}{3}, \quad \pi_1 = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}.$$