

EE 503

Lecture Notes Part 10

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10.0 Two Random Variables

Given two random variables, X and Y , we can regard each pair of outcomes as an element in 2-dimensional space.

10.1 Joint Distribution and Density

Definition: The *joint distribution function*, $F_{XY}(x, y)$, of two random variables, X, Y , is defined by

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

We can write $F(x, y)$ for $F_{XY}(x, y)$ when clear to do so.

Plots of events will be given in class.

Properties:

- i. $P(X \leq x, Y \leq y) \leq P(X \leq x), \quad P(X \leq x, Y \leq y) \leq P(Y \leq y).$
- ii. $P(X = -\infty, Y \leq y) \leq P(X = -\infty) = 0 \Rightarrow F(-\infty, y) = 0.$
- iii. Similarly, $F(x, -\infty) = 0.$
- iv. $P(X \leq \infty, Y \leq \infty) = P(\Omega) = 1 \Rightarrow F(\infty, \infty) = 1.$
- v. $\{x_1 < X \leq x_2, Y \leq y\} = \{X \leq x_2, Y \leq y\} \setminus \{X \leq x_1, Y \leq y\}$ which implies
$$\begin{aligned} P(x_1 < X \leq x_2, Y \leq y) &= P(X \leq x_2, Y \leq y) - P(X \leq x_1, Y \leq y) \\ &= F(x_2, y) - F(x_1, y). \end{aligned}$$
- vi. Similarly,
$$\begin{aligned} P(X \leq x, y_1 < Y \leq y_2) &= P(X \leq x, Y \leq y_2) - P(X \leq x, Y \leq y_1) \\ &= F(x, y_2) - F(x, y_1). \end{aligned}$$
- vii. $\{x_1 < X \leq x_2, y_1 < Y \leq y_2\}$
 $= \{x_1 < X \leq x_2, Y \leq y_2\} \setminus \{x_1 < X \leq x_2, Y \leq y_1\}$
which combined with the above yields
$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1).$$

Definition: The *joint density function*, $f_{XY}(x, y)$, of two random variables, X, Y , is defined by

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}.$$

We can write $f(x, y)$ for $f_{XY}(x, y)$ when clear to do so.

Therefore,

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(u, v) du dv.$$

Recall,

$$F(\infty, \infty) = 1 \Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(u, v) du dv = 1.$$

Definition: In the discrete case we have the *joint probabilities*

$$p_{ik} = P(X = x_i, Y = y_k).$$

Here

$$\sum_{i,k} p_{ik} = 1.$$

Definitions: For two random variables X and Y , $F_X(x)$ is called the *marginal distribution* of X and $f_x(x)$ is called the *marginal density* of X . Similarly, for $F_Y(y)$ and $f_Y(y)$.

Claim:

$$F_X(x) = F(x, \infty), \quad F_Y(y) = F(\infty, y),$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Proof:

$$F_X(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = F(x, \infty).$$

$$F_Y(y) = P(Y \leq y) = P(X \leq \infty, Y \leq y) = F(\infty, y).$$

Now

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v) du dv.$$

So,

$$\frac{\partial F(x, y)}{\partial x} = \int_{-\infty}^y f(x, v) dv, \quad \frac{\partial F(x, y)}{\partial y} = \int_{-\infty}^x f(u, y) du.$$

But,

$$F_X(x) = F(x, \infty) \text{ and } f_X(x) = \frac{d}{dx}F_X(x) = \frac{\partial}{\partial x}F(x, \infty)$$

so

$$f_X(x) = \left. \frac{\partial F(x, y)}{\partial x} \right|_{y=\infty} = \int_{-\infty}^{\infty} f(x, v)dv = \int_{-\infty}^{\infty} f(x, y)dy.$$

Similarly,

$$f_Y(y) = \left. \frac{\partial F(x, y)}{\partial y} \right|_{x=\infty} = \int_{-\infty}^{\infty} f(x, y)dx.$$

Definition: In the discrete case we have *marginal probabilities*, p_i and q_k , where $p_i = P(X = x_i)$ and $q_k = P(Y = y_k)$.

Here

$$p_i = \sum_k p_{ik}, \quad q_k = \sum_i p_{ik}.$$

Example: Suppose the two-dimensional continuous random variable (X, Y) has joint *pdf*

$$f(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Verify this is a valid *pdf*.

Obviously, $f(x, y) \geq 0$. Check

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= 1. \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3} \right) dx dy \\ &= \int_0^2 \left(\frac{x^3}{3} + \frac{x^2 y}{6} \right) \Big|_{x=0}^{x=1} dy = \int_0^2 \left(\frac{1}{3} + \frac{y}{6} \right) dy = \left(\frac{y}{3} + \frac{y^2}{12} \right) \Big|_0^2 = 1. \end{aligned}$$

b. Let $B = \{X + Y \geq 1\}$. Find $P(B)$.

$$P(B) = 1 - P(\bar{B}), \text{ where } \bar{B} = \{X + Y < 1\}.$$

Note $x + y < 1 \Rightarrow y < 1 - x$. So

$$P(B) = 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3} \right) dy dx = \frac{65}{72}.$$

c. Find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3} \right) dy = \int_0^2 \left(x^2 + \frac{xy}{3} \right) dy.$$

Thus,

$$f_X(x) = \begin{cases} 2x^2 + \frac{2x}{3}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Also,

$$f_Y(y) = \int_0^1 \left(x^2 + \frac{xy}{3} \right) dx.$$

We get

$$f_Y(y) = \begin{cases} \frac{1}{3} + \frac{y}{6}, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition: We say (X, Y) is *uniformly distributed* over a region R in the Euclidean plane if

$$f(x, y) = \begin{cases} \frac{1}{\text{area of } R}, & \text{for } (x, y) \in R, \\ 0, & \text{elsewhere.} \end{cases}$$

Example: Suppose (X, Y) is uniformly distributed over the region between the curves $y = x$ and $y = x^2$.

a. Find $f(x, y)$.

$$f(x, y) = \frac{1}{\text{area of } R},$$

where

$$\text{area of } R = \int_0^1 \int_{x^2}^x dy dx = \frac{1}{6}.$$

So

$$f(x, y) = \begin{cases} 6, & \text{for } (x, y) \in R, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find the marginal *pdf's* of X and Y .

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{x^2}^x 6 dy.$$

Thus,

$$f_X(x) = \begin{cases} 6(x - x^2), & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_y^{\sqrt{y}} 6 dx.$$

Thus,

$$f_Y(y) = \begin{cases} 6(\sqrt{y} - y), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

10.2 Independence

Definition: Let (X, Y) be a two-dimensional discrete random variable. We say X and Y are *independent* if $p_{ik} = p_i p_k$, where $p_{ik} = P(X = x_i, Y = y_k)$, $p_i = P(X = x_i)$, $p_k = P(Y = y_k)$, i.e., $P(X = x_1, Y = y_k) = P(X = x_i)P(Y = y_k) \forall i, k$.

Definition: Let (X, Y) be a two-dimensional continuous random variable. We say X and Y are *independent* if $f(x, y) = f_X(x)f_Y(y)$, or equivalently $F(x, y) = F_X(x)F_Y(y)$, or equivalently $P(X \in S_X, Y \in S_Y) = P(X \in S_X)P(Y \in S_Y)$, where S_X and S_Y are arbitrary measurable sets on the x-axis and y-axis, respectively.

Example: From before

$$f(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

We found

$$f_X(x) = \begin{cases} 2x^2 + \frac{2x}{3}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{3} + \frac{y}{6}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Here $f(x, y) \neq f_X(x)f_Y(y)$ so X and Y are not independent.

Theorem: Let (X, Y) be a two-dimensional random variable. Let A and B be events whose occurrence (or nonoccurrence) depends only on X and Y , respectively. (That is, $A \subset R_X$, $B \subset R_Y$.) Then, if X and Y are independent random variables, $P(A \cap B) = P(A)P(B)$.

Proof: (continuous case only).

$$\begin{aligned} P(A \cap B) &= \int \int_{A \cap B} f(x, y) dx dy = \int \int_{A \cap B} f_X(x) f_Y(y) dx dy \\ &= \int_A f_X(x) dx \int_B f_Y(y) dy \\ &= P(A)P(B). \end{aligned}$$

10.3 One Function of Two Random Variables

Given random variables X and Y and a function $g(x, y)$, we form the random variable

$$Z = g(X, Y).$$

We find the distribution function as

$$F_Z(z) = P(Z \leq z) = P(g(X, Y) \leq z) = P((x, y) \in D_z)$$

where, D_z is the region in the xy -plane consisting of the set

$$\{(x, y) : x \in R_X, y \in R_Y, g(x, y) \leq z\}.$$

So,

$$F_Z(z) = \int \int_{D_z} f(x, y) dx dy$$

where, $f(x, y)$ is the joint density of X and Y .

Example: Given X and Y with joint *pdf*

$$f(x, y) = \begin{cases} e^{-(x+y)}, & x \geq 0, y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

- a. Are X and Y independent?
 b. Find the distribution function and the *pdf* of $Z = \min \{X, Y\}$.

Solution:

- a. X and Y independent if and only if $f(x, y) = f_X(x)f_Y(y)$. Now

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{\infty} e^{-(x+y)} dy$$

or

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} e^{-y}, & y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

So $f(x, y) = e^{-(x+y)} = e^{-x}e^{-y} = f_X(x)f_Y(y) \Rightarrow X$ and Y are independent.

- b. $Z = \min \{X, Y\}$. Now

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(\min \{X, Y\} \leq z) \\ &= 1 - P(\min \{X, Y\} > z) \\ &= 1 - P(X > z, Y > z) = 1 - P(X > z)P(Y > z). \end{aligned}$$

Now

$$F_X(x) = \int_{-\infty}^x f_X(u)du = \int_0^x e^{-u} du = (1 - e^{-x}) U(x).$$

Similarly,

$$F_Y(y) = (1 - e^{-y}) U(y).$$

So

$$F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z)) = (1 - e^{-z}e^{-z}) U(z)$$

or

$$F_Z(z) = \begin{cases} 1 - e^{-2z}, & z \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, $f_Z(z) = \frac{d}{dz}F_Z(z)$ or

$$f_Z(z) = \begin{cases} 2e^{-2z}, & z \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Example: Suppose that X and Y are independent random variables each uniformly distributed over $[0, 1]$. Let $Z = X + Y$. Find the *pdf* of Z , $f_Z(z)$, and the distribution function, $F_Z(z)$.

Solution: The joint *pdf* is

$$f(x, y) = f_X(x)f_Y(y),$$

where

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Now,

$$F_Z(z) = P(Z \leq z) = P(X + Y \leq z) = \int \int_{(x,y):x+y \leq z} f(x, y) dx dy.$$

There are four cases to consider:

1. $z < 0$. Set $x + y = z \Rightarrow y = -x + z$. Here $F_Z(z) = 0$.

2. $0 \leq z \leq 1$.

$$F_Z(z) = \int_0^z \int_0^{-x+z} dy dx = z^2/2.$$

3. $1 < z \leq 2$.

$$F_Z(z) = \int_0^{z-1} \int_0^1 dy dx + \int_{z-1}^1 \int_0^{-x+z} dy dx = 2z - z^2/2 - 1.$$

4. $z > 2$. Here $F_Z(z) = 1$.

So

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ z^2/2, & 0 \leq z \leq 1, \\ 2z - z^2/2 - 1, & 1 < z \leq 2, \\ 1, & z > 2. \end{cases}$$

Definition: Two random variables X and Y are said to be *jointly normal* if

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \cdot \exp \left[\frac{-1}{2(1-r^2)} \left(\frac{(x-\mu_X)^2}{\sigma_X^2} - 2r \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right) \right].$$

Here $|r| < 1$ where r is called the correlation coefficient (more about this later). Integration shows

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \cdot \exp \left(-\frac{(x-\mu_X)^2}{2\sigma_X^2} \right)$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \cdot \exp \left(-\frac{(y-\mu_Y)^2}{2\sigma_Y^2} \right).$$

If $r = 0$ then X and Y are said to be uncorrelated and in the normal case they are also independent.

Example: Let

$$f(x, y) = \frac{1}{2\pi\sigma^2} \cdot \exp \left(-\frac{x^2 + y^2}{2\sigma^2} \right).$$

Here $f(x, y) = f_X(x)f_Y(y)$ where

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp \left(-\frac{x^2}{2\sigma^2} \right)$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp \left(-\frac{y^2}{2\sigma^2} \right).$$

Find $f_Z(z)$ where $Z = \sqrt{X^2 + Y^2}$.

Solution: Here $Z^2 = X^2 + Y^2$. Let

$$x = r \cos \theta, \quad y = r \sin \theta, \quad dx dy = r dr d\theta.$$

Then,

$$F_Z(z) = \int \int_{D_z} f_{XY}(x, y) dx dy = \int \int_{D_z} \frac{1}{2\pi\sigma^2} \cdot \exp \left(-\frac{x^2 + y^2}{2\sigma^2} \right)$$

$$= \int \int_{D_z^*} f_{R\Theta}(r, \theta) dr d\theta$$

where

$$D_z = \left\{ (x, y) : \sqrt{x^2 + y^2} \leq z \right\}, \quad D_z^* = \{r, \theta\} : 0 < r \leq z, 0 < \theta \leq 2\pi\}.$$

So

$$\begin{aligned} F_Z(z) &= \frac{1}{2\pi\sigma^2} \int_0^z \int_0^{2\pi} r e^{-r^2/2\sigma^2} d\theta dr \\ &= 1 - e^{-z^2/2\sigma^2}, \quad z \geq 0. \end{aligned}$$

Thus,

$$f_Z(z) = \begin{cases} \frac{z}{\sigma^2} \cdot e^{-z^2/2\sigma^2}, & z \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

This is a *Rayleigh* density.

10.4 Two Functions of Two Random Variables

Here we have two random variables X and Y and form

$$Z = g(X, Y), \quad W = h(X, Y).$$

Now

$$\begin{aligned} F_{ZW}(z, w) &= P(Z \leq z, W \leq w) = P(g(X, Y) \leq z, h(X, Y) \leq w) \\ &= P((X, Y) \in D_{zw}) \end{aligned}$$

where, D_{zw} is the region in the xy -plane consisting of the set

$$\{(x, y) : x \in R_X, y \in R_Y, g(x, y) \leq z, h(x, y) \leq w\}.$$

So,

$$F_{ZW}(z, w) = \int \int_{D_{zw}} f_{XY}(x, y) dx dy.$$

Theorem: Let X and Y be two random variables and let $Z = g(X, Y)$ and $W = h(X, Y)$. Let $g(x, y) = z$ and $h(x, y) = w$ have solutions (x_n, y_n) , $n = 1, 2, \dots$. Then

$$f_{ZW}(z, w) = \frac{f_{XY}(x_1, y_1)}{|J(x_1, y_1)|} + \frac{f_{XY}(x_2, y_2)}{|J(x_2, y_2)|} + \dots$$

where J is the Jacobian defined as

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1}$$

Proof: See posted handout.

Example: Suppose we throw a dart at a unit circle. Let (x, y) denote the coordinates where it lands. The area of the unit circle is $\pi r^2 = \pi$ so assume

$$f_{XY}(x, y) = \begin{cases} 1/\pi, & (x, y) \text{ inside circle,} \\ 0, & \text{elsewhere.} \end{cases}$$

Define $R = \sqrt{X^2 + Y^2}$. Find $f_R(r)$.

Solution: Let $\Phi = \tan^{-1}(Y/X)$. Then

$$R = g(X, Y), \text{ where } g(x, y) = \sqrt{x^2 + y^2}$$

and

$$\Phi = h(X, Y), \text{ where } h(x, y) = \tan^{-1}(y/x).$$

We solve for x, y as

$$x = r \cos \phi, \quad y = r \sin \phi.$$

Thus,

$$J(r, \phi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix}^{-1} = \begin{vmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{vmatrix}^{-1} = \frac{1}{r}.$$

So,

$$f_{R\Phi}(r, \phi) = f_{XY}(r \cos \phi, r \sin \phi) r = r/\pi, \quad 0 \leq r \leq 1, \quad 0 \leq \phi < 2\pi$$

and

$$f_R(r) = \int_0^{2\pi} f_{R\Phi}(r, \phi) d\phi = \int_0^{2\pi} \frac{r}{\pi} d\phi = 2r, \quad 0 \leq r \leq 1.$$

Theorem: Let (X, Y) be a continuous two-dimensional random variable and assume X and Y are independent. Let $W = XY$. Then

$$f_W(w) = \int_{-\infty}^{\infty} f_X(z) f_Y(w/z) \left| \frac{1}{z} \right| dz.$$

Proof: $w = xy$. Let $z = x$. Thus, $x = z$ and $y = w/z$.

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1} = \begin{vmatrix} 1 & 0 \\ -w/z^2 & 1/z \end{vmatrix}^{-1} = z.$$

So,

$$f_{ZW}(z, w) = f_{XY}(z, w/z) \left| \frac{1}{z} \right| = f_X(z) f_Y(w/z) \left| \frac{1}{z} \right|.$$

Thus,

$$f_W(w) = \int_{-\infty}^{\infty} f_X(z) f_Y(w/z) \left| \frac{1}{z} \right| dz.$$

Theorem: Let (X, Y) be a continuous two-dimensional random variable and assume X and Y are independent. Let $Z = X/Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(wz) f_Y(w) |w| dw.$$

Proof: $z = x/y$. Let $w = y$. Thus, $x = wz$ and $y = w$.

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix}^{-1} = \frac{1}{w}.$$

So,

$$f_{ZW}(z, w) = f_X(wz) f_Y(w) |w|.$$

Thus,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(wz) f_Y(w) |w| dw.$$

Theorem: Let (X, Y) be a continuous two-dimensional random variable. Let $Z = X + Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z - w) dw.$$

Proof: $z = x + y$. Let $w = x$. Thus, $y = z - w$ and $x = w$.

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}^{-1} = -1.$$

So,

$$f_{ZW}(z, w) = f_{XY}(w, z - w).$$

Thus,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z-w)dw.$$

Special Case: If X and Y are independent then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w)f_Y(z-w)dw = f_X * f_Y.$$

We can deduce this last result a more direct way as follows: Recall X and Y are independent and $Z = X + Y$.

$$F_Z(z) = P(Z \leq z) = P((X + Y) \leq z) = \int \int_{D_z} f_X(x)f_Y(y)dxdy$$

where

$$D_z = \{(x, y) : x + y \leq z\}.$$

Thus

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x)f_Y(y)dydx \\ &= \int_{-\infty}^{\infty} f_X(x) \left[\int_{-\infty}^{z-x} f_Y(y)dy \right] dx. \end{aligned}$$

Then

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

Example: Suppose T_1 and T_2 represent the life lengths of electronic components with *pdf*

$$f_{T_1}(t_1) = \begin{cases} \alpha_1 e^{-\alpha_1 t_1}, & t_1 \geq 0, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_{T_2}(t_2) = \begin{cases} \alpha_2 e^{-\alpha_2 t_2}, & t_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Suppose component 2 turns on when component 1 ceases operation. Find the *pdf* of the total life length $T = T_1 + T_2$.

Solution:

i. Assume $\alpha_1 \neq \alpha_2$. Then

$$f_T(t) = \int_{-\infty}^{\infty} f_{T_1}(t_1)f_{T_2}(t-t_1)dt_1, \quad t \geq 0.$$

Now $t_1 \geq 0$ (due to f_{T_1}) and $(t-t_1) \geq 0$ (due to f_{T_2}). Thus $0 \leq t_1 \leq t$.

So,

$$f_T(t) = \int_0^t \alpha_1 e^{-\alpha_1 t_1} \alpha_2 e^{-\alpha_2(t-t_1)} dt_1$$

or

$$f_T(t) = \begin{cases} \frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} (e^{-\alpha_1 t_1} - e^{-\alpha_2 t_2}), & t \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

ii. Assume $\alpha_1 = \alpha_2 = \alpha$. Then

$$f_T(t) = \int_0^t \alpha e^{-\alpha t_1} \alpha e^{-\alpha(t-t_1)} dt_1$$

or

$$f_T(t) = \begin{cases} \alpha^2 t e^{-\alpha t}, & t \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Linear Transformations

Let $Z = aX + bY$, $W = cX + dY$ for constants a, b, c, d . Then the values these random variables can take on can be written as

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we let M be the 2×2 matrix above then if M^{-1} exists, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} z \\ w \end{bmatrix}.$$

Then $x = Az + Bw$, $y = Cz + Dw$ for some constants A, B, C, D . Now

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

So,

$$f_{ZW}(z, w) = \frac{1}{|ad - bc|} f_{XY}(Az + Bw, Cz + Dw).$$

Theorem: Let X and Y be two independent random variables each of which may assume only nonnegative integral values. Let

$$p_k = P(X = k), k = 0, 1, 2, \dots$$

and

$$q_r = P(Y = r), r = 0, 1, 2, \dots$$

Let $W = X + Y$ and let $w_i = P(W = i)$. Then

$$w_i = \sum_{k=0}^i p_k q_{i-k}, \quad i = 0, 1, 2, \dots$$

Proof: $w_i = P(w = i)$ so

$$\begin{aligned} w_i &= P(X = 0, Y = i \text{ or } X = 1, Y = i - 1 \text{ or } \dots \text{ or } X = i, Y = 0) \\ &= \sum_{k=0}^i P(X = k, Y = i - k) = \sum_{k=0}^i P(X = k)P(Y = i - k) = \sum_{k=0}^i p_k q_{i-k}. \end{aligned}$$