EE 503

Lecture Notes Part 5

Christopher Wayne Walker, Ph.D.

5.0 Conditional Probability

5.1 Definitions

Often we are interested in calculating the likelihood of an event occurring given that another event occurs.

Definition: The *conditional probability* of event B occurring given that event A occurs is defined to be

$$P(B|A) = \begin{cases} \dfrac{P(B \cap A)}{P(A)}, & P(A) > 0 \\ 0, & P(A) = 0. \end{cases}$$

Note: In a finite sample space model with equally likely outcomes

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} = \frac{\frac{\text{number of elements in } (B \cap A)}{\text{number of elements in } \Omega}}{\frac{\text{number of elements in } A}{\text{number of elements in } \Omega}} \\ &= \frac{\text{number of elements in } (B \cap A)}{\text{number of elements in } A} \end{aligned}$$

which gives some justification to the definition of conditional probability shown above.

Now

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B|A)P(A).$$

Interchanging A and B gives

$$P(A \cap B) = P(A|B)P(B).$$

So

$$P(B|A) = \begin{cases} \dfrac{P(A|B)P(B)}{P(A)}, & P(A) > 0 \\ 0, & P(A) = 0. \end{cases}$$

5.2 Properties of Conditional Probabilities

Let Ω be a sample space and assume $P(A) > 0$. Then

i. $P(\Omega|A) = 1$.

ii. $P(B|A) \geq 0$ for any event B.

iii. If $B_1,\ B_2, \ldots,\ B_k$ are mutually exclusive events, i.e., $B_i \cap B_j = \emptyset$ for $i \neq j$ then

$$P(B_1 \cup B_2 \cup \cdots \cup B_k|A) = P(B_1|A) + P(B_2|A) + \cdots + P(B_k|A).$$

iv. $P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$.

Given a partition $B_1,\ B_2, \ldots, B_k$ of the sample space Ω, i.e., events $B_1,\ B_2, \ldots, B_k$ such that $P(B_i) > 0$, $i = 1, 2, \ldots, k$, $B_i \cap B_j = \emptyset$, $i \neq j$ and $\bigcup_{i=1}^{k} B_i = \Omega$ then

v. (law of total probability)

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_k)P(B_k)$$

and

vi. (Bayes' Rule)

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_k)P(B_k)}.$$

Proof: We will prove (v) and (vi) above.

$$\begin{aligned} A &= A \cap \Omega = A \cap (B_1 \cup B_2 \cup \cdots \cup B_k) \\ &= (A \cap B_1) \cup \cdots \cup (A \cap B_k). \end{aligned}$$

Thus,

$$P(A) = P(A \cap B_1) \cup \cdots \cup P(A \cap B_k)$$

so

$$P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_k)P(B_k)$$

which proves (v). Now

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{P(A|B_1)P(B_1) + \cdots + P(A|B_k)P(B_k)}$$

which proves (vi).