EE 503

Lecture Notes Part 4

Christopher Wayne Walker, Ph.D.

4.0 Combinatorics

Here we are concerned with

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega} = \frac{n_A}{n}.$$

We will study techniques for finding n_A and n. It is understood that if we choose an object at random from a collection of n objects, then each object has the same likelihood of being chosen. More generally, if we choose k objects from n objects then each k-tuple is equally likely.

Multiplication Principle:

Suppose operation 1, O_1, can be performed in n_1 ways, operation 2, O_2, can be performed in n_2 ways, ..., operation k, O_k, can be performed in n_k ways. Suppose O_n follows O_{n-1}, $n = 1, 2, \ldots, k$. The total number of ways of performing all k operations is $n_1 n_2 \cdots n_k$.

Addition Principle:

Again we perform operation O_n in n ways, $n = 1, 2, \ldots, k$. The total number of ways of performing O_1 or O_2 or $\cdots$ or O_k is $n_1 + n_2 \cdots + n_k$.

Permutations:

i. Arrange n different objects.

   Definition: ${}_nP_n$ is the total number of ways of arranging or permuting n different objects. Note:

   $${}_nP_n = n!.$$

ii. Pick r of n objects and permute these.

   Definition: ${}_nP_r$ is the total number of ways of permuting r of n objects. Note:

   $${}_nP_r = \frac{n!}{(n-r)!}.$$

Combinations:

Pick r of n objects but do not care about order.

Definition: C is the number of ways of choosing r of n objects disregarding order. From the last result we know

$$Cr! = \frac{n!}{(n-r)!}.$$

So

$$C = \frac{n!}{(n-r)!r!}.$$

Notation: C is usually written as $\binom{n}{r}$. Thus,

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

$\binom{n}{r}$ are called *binomial coefficients* since they appear as coefficients in the expansion of the binomial expression $(a+b)^n$. This leads to the *binomial theorem*:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Permutations when not all objects are different: Recall we can arrange n different objects in $n!$ ways. Say, we have n_1 of one kind, n_2 of another,..., n_k of the kth kind such that $n_1 + n_2 + \cdots + n_k = n$. As will be explained in class, the number of permutations of these n objects is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Example: Tickets are numbered 1 to 100. Choose 3 tickets without replacement. Find $P(A)$ where A is the event

$$A = \{\text{all 3 tickets are numbered from 1 to 10}\}.$$

Here

$$\Omega = \{(x_1,\ x_2,\ x_3) : \text{each } x_i \text{ is from 1 to 100 and all } x_i's \text{ are different}\}.$$

i. Consider without replacement, without order.

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega} = \frac{\binom{10}{3}}{\binom{100}{3}} = 0.000742.$$

We worked this problem without replacement (required) and without order (optional).

ii. Consider without replacement, with order.

$$P(A) = \frac{{}_{10}P_3}{{}_{100}P_3} = 0.000742.$$

So we can work some problems different ways.

Other examples will be given in class.