EE 503

Lecture Notes Part 3

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3.0 Probability Measure

3.1 Sample Space and Events

Definition: The *sample space* Ω is the set of all possible outcomes of a random experiment.

Definition: An *event* is a (particular kind of) subset of the sample space (this will be clarified later by showing that not all subsets are allowed to be events).

Examples:

i. Choose an integer between 1 and 5. Then

$$\Omega = \{1,\ 2,\ 3,\ 4,\ 5\}.$$

We see that $\{2,\ 4\}$ is the event that an even number was chosen.

ii. Choose a real number in the interval $[0,\ 1]$. Then

$$\Omega = [0,\ 1].$$

Note that Ω is uncountable (cannot be put into a 1-1 correspondence with the integers) and not every subset is an event (we will prove this later). But any open or closed subset of $[0,\ 1]$ is an event, in particular, any subinterval of $[0,\ 1]$ is an event.

Rules for Events

i. $\emptyset$ and Ω are events.

ii. If $\mathbf{A}$ and $\mathbf{B}$ are events then so are $\mathbf{A} \cap \mathbf{B}$, $\mathbf{A} \cup \mathbf{B}$, $\mathbf{B} \backslash \mathbf{A}$ and $\overline{\mathbf{A}} = \Omega \backslash \mathbf{A}$ (note: $b \in \mathbf{B} \backslash \mathbf{A}$ *iff* $b \in \mathbf{B}$ but $b \notin \mathbf{A}$). $\mathbf{B} - \mathbf{A}$ is also written for $\mathbf{B} \backslash \mathbf{A}$.

iii. If $A_1,\ A_2, \ldots,$ are events then so are

$$A_1 \cup A_2 \cup \cdots = \bigcup_{n=1}^{\infty} \mathbf{A}_n \quad \text{and} \quad A_1 \cap A_2 \cap \cdots = \bigcap_{n=1}^{\infty} \mathbf{A}_n.$$

Remarks

i. $\overline{\Omega} = \emptyset$.

ii. $\mathbf{A} \cup \mathbf{B} \cup \emptyset \cup \emptyset \cup \cdots = \mathbf{A} \cup \mathbf{B}$.

iii. $\mathbf{A} \cap \mathbf{B} = \Omega \backslash \left(\overline{\mathbf{A}} \cup \overline{\mathbf{B}} \right) = \overline{\overline{\mathbf{A}} \cup \overline{\mathbf{B}}}$.

iv. $\mathbf{A} \backslash \mathbf{B} = \mathbf{A} \cap \overline{\mathbf{B}}$.

v. $\mathbf{A} \triangle \mathbf{B} := (\mathbf{A} \backslash \mathbf{B}) \cup (\mathbf{B} \backslash \mathbf{A})$ (this is the definition of symmetric difference).

Definition: Let $\mathbf{X}$ be a nonempty set. An *algebra* of sets on $\mathbf{X}$ is a nonempty collection $A \in P(\mathbf{X})$ which is closed under finite unions and complements, i.e.,

i. if $E_1,\ E_2, \ldots,\ E_n \in A$ then $\bigcup\limits_{k=1}^{n} E_k \in A$.

ii. if $E \in A$ then $\overline{E} \in A$.

Definition: A σ-*algebra* (or σ-*field*) is an algebra which is closed under countable unions. So, a collection F of subsets of $\mathbf{X}$ is called a σ-field if

i. $\emptyset \in F$.

ii. $A_1,\ A_2, \ldots \in F \Rightarrow \bigcup\limits_{k=1}^{\infty} A_k \in F$.

iii. $A \in F \Rightarrow \overline{A} \in F$.

Definition: A *metric* d on a nonempty set $\mathbf{X}$ is a function $d : \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ satisfying

i. $d(x, y) \geq 0 \ \forall\ x, y \in \mathbf{X}$ and $d(x, y) = 0 \Rightarrow x = y$.

ii. $d(x, y) = d(y, x) \ \forall\ x, y \in \mathbf{X}$.

iii. $d(x, y) \leq d(x, z) + d(z, y) \ \forall\ x, y, z \in \mathbf{X}$ (triangle inequality).

The pair $(\mathbf{X}, d)$ is called a *metric space*.

Definition: Let $(\mathbf{X}, d)$ be a metric space. If $x \in \mathbf{X}$ and $r > 0$, the *open ball* of radius r about x is

$$B(r, x) = \{y \in \mathbf{X} : d(x, y) < r\}.$$

Definition: A set $E \subset \mathbf{X}$ is *open* if for every $x \in E$ there exists $r > 0$ such that $B(r, x) \subset E$.

Definition: A set is *closed* if its complement is open.

Definition: If $(\mathbf{X}, d)$ is any metric space, the σ-field generated by the family of open sets in $\mathbf{X}$ is called the *Borel* σ-field, denoted by $B_{\mathbf{X}}$. Its members are called Borel sets.

Special case: When $\mathbf{X} = \mathbf{R}$ (the real line), we have $B_{\mathbf{R}}$.

3.2 Probability Space

Let Ω be a sample space with F a σ-field. We wish to assign a number $P(A)$ to each event A representing the likelihood the event will occur.

Consider two motivations for $P(A)$.

i. Ω is finite with all outcomes equally likely. Take

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}.$$

Note: $P(\emptyset) = 0, \quad P(\Omega) = 1, \quad 0 \leq P(A) \leq 1$. If A and B are disjoint, i.e., $A \cap B = \emptyset$, then

$$\begin{aligned} P(A \cup B) &= \frac{\text{number of elements in } A + \text{number of elements in } B}{\text{number of elements in } \Omega} \\ &= P(A) + P(B). \end{aligned}$$

ii. Ω is countably infinite. Repeat the experiment many times. Take

$$P(A) = \lim_{n \to \infty} \frac{\text{number of occurrences of } A \text{ in } n \text{ trials}}{n}.$$

Definition: A *probability measure* P on (Ω, F) is a function

$$P : F \to [0,\ 1]$$

that maps

$$A \mapsto P(A)$$

that satisfies the following axioms of probability:

i. $P(\emptyset) = 0, \quad P(\Omega) = 1, \quad 0 \le P(A) \le 1$ (redundant)

ii. If $A_1,\ A_2, \ldots$ is a sequence of pairwise disjoint events then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

.

The triple $(\Omega,\ F,\ P)$ is called a *probability space.*

Remark: If we take $A_3 = A_4 = \cdots = \emptyset$ in (ii) above we get

ii′. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.

(ii′) is called *finite additivity.* (ii) is called *countable additivity.*

Some consequences of the above will be presented in class.

Examples of Probability Spaces:

i. Ω is finite with all outcomes equally likely. $F = \sigma$-field= all subsets of Ω. Take

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}.$$

ii. Ω is finite, $\Omega = \{\omega_1,\ \omega_2, \ldots,\ \omega_n\}$, $F = \sigma$-field= all subsets of Ω. Say, $P(\{\omega_i\}) = P(\omega_i) = p_i$, where $p_1,\ p_2, \ldots,\ p_n$ satisfy $p_i \ge 0$ for all i and $p_1 + p_2 + \cdots + p_n = 1$. Take

$$P(A) = \sum_{i:\omega_i \in A} p_i.$$

iii. Ω is countably infinite, $\Omega = \{\omega_1,\ \omega_2, \ldots\}$, $F = \sigma$-field= all subsets of Ω. Say, $P(\{\omega_i\}) = p_i$, where $p_1,\ p_2, \ldots$ satisfy $p_i \geq 0$ for all i and $p_1 + p_2 + \cdots = 1$. Take

$$P(A) = \sum_{i:\omega_i \in A} p_i.$$

iv. $\Omega = [0,\ 1]$. Let $A \subset \Omega$. It is sensible to take

$$P(A) = \frac{\text{length of } A}{\text{length of } \ \Omega}.$$

This is okay for A consisting of a collection of intervals but not for all subsets of Ω. We can find a "bad" A (will show later).