EE 503

Quiz 14 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (7 points.) Let $X = (X_1, X_2, \ldots, X_n)$ be i.i.d. where each $X_i \sim U(-\theta, \theta)$ where θ is unknown. Find the MLE for θ.

Solution: We find the density for X is

$$f(x) = \begin{cases} \left(\frac{1}{2\theta}\right)^n, & -\theta < X_{(1)} < X_{(n)} < \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

To maximize $f(x)$ we want to make $\hat{\theta}$ as small as possible. We need $X_{(n)} < \theta$ and $-X_{(1)} < \theta$. Hence, we choose

$$\hat{\theta} = \max\{X_{(n)}, -X_{(1)}\}$$

or equivalently,

$$\hat{\theta} = \max\{X_i\}, \; i = 1, 2, \ldots, n.$$

Problem 2. (8 points.) Let $X = (X_1, X_2, \ldots, X_n)$ be i.i.d. where each X_i is exponentially distributed with parameter $\lambda > 0$, that is, each X_i has probability density function

$$f(x_i) = \lambda e^{-\lambda x_i}, \quad x_i \geq 0$$

and is zero elsewhere.

a. Find the MLE for λ.

   Solution: We find the density for X is

   $$f(x) = \begin{cases} \lambda^n \exp(-\lambda \sum_i x_i), & \text{all } x_i \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

   and thus

   $$\log f(x) = n \log \lambda - \lambda \sum_i x_i.$$

   Taking the derivative w.r.t. λ abd setting equal to 0 yields

   $$\hat{\lambda} = \frac{n}{\sum_i x_i} = \frac{1}{\bar{X}}.$$

b. Compute the Cramer-Rao inequality for any unbiased estimator of λ.

Solution: Since we have an i.i.d. exponential family we find for an unbiased estimator $W(\mathbf{X})$

$$\begin{aligned} Var_\lambda W(\mathbf{X}) &\geq \frac{1}{-nE_\lambda\left(\frac{\partial^2}{\partial\lambda^2}\log f(X|\mu)\right)} \\ &= \frac{\lambda^2}{n}. \end{aligned}$$