EE 503

Quiz 12 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (7 points.) Choose ω uniform in the interval $[0,\ 1]$. For $n = 1, 2, \ldots$, define

$$X_n(\omega) = \begin{cases} 0, & \omega \geq \dfrac{n}{2n+1}, \\ 1, & \text{elsewhere}, \end{cases}$$

$$X(\omega) = \begin{cases} 1, & \omega < 1/2, \\ 0, & \text{elsewhere}. \end{cases}$$

Determine if X_n converges to X surely, almost surely, or neither.

Solution: $X_n \to 0$ surely.

Problem 2. (8 points.) Let $X_k,\ k = 1, 2, \ldots$ be a sequence of independent normal (Gaussian) random variables with mean 0 and variance σ^2. Let Y be a normal random variable with mean 0 and variance $\frac{1}{2}$. Define

$$Y_n = (1-\alpha)X_n + \alpha Y_{n-1}, \quad n = 1, 2 \ldots$$

where we take $Y_0 = 0$ and $\alpha \in (0, 1)$.

a. Determine the value of $\alpha \in (0, 1)$ such that Y_n converges to Y in distribution. Justify your answer.

   Solution: Since we are concerned with convergence in distribution involving Gaussian random variables it is enough to show that the mean and variance of Y_n matches that of Y as $n \to \infty$. It is easy to see that $E[Y_n] = 0$ for all n. For variance we compute

   $$Y_n = (1-\alpha)X_n + \alpha Y_{n-1} = (1-\alpha)\sum_{k=1}^{n} \alpha^{n-k} X_k$$

   so

$$
\begin{aligned}
Var(Y_n) &= (1-\alpha)^2 \sum_{k=1}^{n} \left(\alpha^{n-k}\right)^2 \sigma^2 \\
&= (1-\alpha)^2 \alpha^{2n} \sigma^2 \frac{\alpha^{-2} - \alpha^{-2(n+1)}}{1-\alpha^{-2}} \\
&= (1-\alpha)^2 \sigma^2 \frac{\alpha^{2n-2} - \alpha^{-2}}{1-\alpha^{-2}} \\
&\to (1-\alpha)^2 \sigma^2 \frac{1}{1-\alpha^2} \\
&= \frac{1-\alpha}{1+\alpha} \cdot \sigma^2.
\end{aligned}
$$

Thus, we solve

$$\frac{1-\alpha}{1+\alpha} \cdot \sigma^2 = \frac{1}{2}$$

to get

$$\alpha = \frac{2\sigma^2 - 1}{2\sigma^2 + 1}$$

provided $\sigma^2 > \frac{1}{2}$. If $\sigma^2 \leq \frac{1}{2}$ then no solution exists.

b. Can Y_n converge to Y in probability? Explain why or why not.

Solution: No. Convergence in probability requires the value that the random variable Y_n takes on is closely connected (in probability) to the value of Y. This does not happen here. They simply have the same distribution as $n \to \infty$.