EE 503

Quiz 11 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (8 points.) Suppose ω is selected at random in the interval [0, 1]. For each of the following state whether the sequence of random variables converges surely, almost surely or not at all. If the sequence does converge indicate the random variable or constant to which the sequence converges.

a. $V_n(\omega) = \omega^n$.

Solution: $V_n \to 0$ almost surely.

b. $W_n(\omega) = \cos^n 2\pi\omega$.

Solution: $W_n \to 0$ almost surely.

c. $Y_n(\omega) = \omega \left(1 + \frac{1}{n}\right)^n$.

Solution: $Y_n \to \omega e$ surely.

d. $X_n(\omega) = \omega^{1/n}$.

Solution: $X_n \to 1$ almost surely.

Problem 2. (7 points.)

a. Give an example of a sequence of random variables, X_n, that converges everywhere (or surely) to a constant, x_0, but does not converge in mean square to x_0.

   Solution: Let X be chosen from a Cauchy distribution. Let $X_n = \frac{X}{n}$. Then X_n converges to 0 surely but does not converge to 0 in mean square since the Cauchy random variable does not have any defined moments. Other solutions are possible.

b. Consider the following random sequence: Set $X_1 = 1$ and set exactly one of X_2, X_3 equal to 1 and set the other to 0 (equally likely), then set exactly one of X_4, X_5, X_6 equal to 1 and set the other two to 0 (equally likely), then set exactly one of X_7, X_8, X_9, X_{10} equal to 1 and set the other three to 0 (equally likely) and continue this process.

   i. Does this sequence converge to 0 in probability? Justify your answer.

      Solution: Yes. Let $\epsilon > 0$. Then, $P(|X_n| \geq \epsilon) \to 0$ as $n \to \infty$ since the nonzero values become further and further separated.

   ii. Does this sequence converge to 0 in mean square? Justify your answer.

      Solution: Yes. Similar reason as above.

   iii. Does this sequence converge to 0 with probability 1 (or almost surely)? Justify your answer.

      Solution: No. Since as n becomes large there are always an infinite number of m values greater than n where X_m will be 1 and not zero. So, $P(|X_n| \geq \epsilon \text{ i.o.})$ is not 0.