EE 503

Quiz 9 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (8 points.) Let W be a Bernoulli random variable with $P(W = 1) = p$ and $P(W = 0) = 1 - p$ and let Z be a random variable having an exponential density of the form $f_Z(z) = \lambda e^{-\lambda z} U(z)$. W and Z are independent. Define $X = W$, $Y = WZ$. Find

a. Find $E\left(E(Y|X)\right)$.

Solution: We find

$$E(Y|X = x) = E(WZ|W = x) = E(xZ) = \frac{x}{\lambda}$$

so

$$E(Y|X) = E(WZ|W = x) = \frac{X}{\lambda}$$

so

$$E\left(E(Y|X)\right) = \frac{p}{\lambda}.$$

b. Find $E\left(Var(Y|X)\right)$.

Solution: We find

$$Var(Y|X = x) = Var(WZ|W = x) = Var(xZ) = \frac{x^2}{\lambda^2}$$

so

$$Var(Y|X) = \frac{X^2}{\lambda^2}$$

so

$$E\left(Var(Y|X)\right) = \frac{p}{\lambda^2}.$$

c. Find the best MSE predictor of Y based on the random variable X.

Solution: We find

$$E(Y|X) = \frac{X}{\lambda}.$$

d. Find the best linear MSE predictor of Y based on X.

Solution: We find the best linear MSE predictor is $a_1X + b_1$ where

$$\begin{aligned} a_1 &= \frac{Cov(X,Y)}{Var(X)} = \frac{E(XY) - E(X)E(Y)}{Var(X)} \\ &= \frac{E(W^2Z) - E(W)E(WZ)}{Var(W)} = \frac{E(W^2)E(Z) - E(W)E(W)E(Z)}{Var(W)} \\ &= \frac{\frac{p}{\lambda} - \frac{p^2}{\lambda}}{pq} = \frac{1}{\lambda}. \end{aligned}$$

$$\begin{aligned} b_1 &= E(Y) - a_1E(X) = E(WZ) - a_1E(W) = E(W)E(Z) - a_1E(W) \\ &= \frac{p}{\lambda} - \frac{p}{\lambda} = 0. \end{aligned}$$

Thus,

$$\hat{Y} = \frac{X}{\lambda}.$$

Problem 2. (7 points.)

a. Consider the random variable X with the Pareto density

$$f(x) = \begin{cases} \lambda x^{-\lambda-1}, & x > 1,\ \lambda > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = \ln(X)$ (the natural log). Find the density function for Y.

Solution: We find

$$F_Y(y) = P(Y \leq y) = P(\ln(X) \leq y) = P(X \leq e^y) = F_X(e^y).$$

Now

$$F_X(x) = \int_1^x \lambda t^{-\lambda-1} dt = 1 - x^{-\lambda}.$$

Thus,

$$F_Y(y) = 1 - e^{-xy}$$

and then

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Consider the random variables X and Y with joint density

$$f(x, y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y}, & x > 0,\ y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Compute $E[X|Y = y]$.

Solution: We compute

$$E[X|Y = y] = \int_0^\infty xf(x|y)dx = \int_0^\infty x\frac{f(x, y)}{f(y)}dx$$

Now

$$f(y) = \int_0^\infty f(x, y)dx = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}dx = e^{-y}$$

so

$$E[X|Y = y] = \int_0^\infty \frac{x}{y}e^{-x/y}dx = y.$$