EE 503

Quiz 8 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (7 points.) Suppose the two-dimensional ranndom variable (X, Y) has density

$$f(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 < x < 1,\ 0 < y < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Compute $P(X + Y \leq 1)$.

Solution: We find

$$\begin{aligned} P(X+Y \leq 1) &= P(Y \leq 1 - X) = \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3}\right) dy dx \\ &= \frac{7}{72} = 0.0972. \end{aligned}$$

Problem 2. (8 points.)

a. Suppose X and Y are independent random variables with X having pdf $f_X(x) = \frac{1}{2}x^2e^{-x}$, $x > 0$ and 0 elsewhere, and Y having pdf $f_Y(y) = 2y$, $0 < y < 1$ and 0 elsewhere. Let $Z = XY$. Find the pdf of Z.

   Solution: Using a theorem in class for a product or random variables we find

   $$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z/w) \left|\frac{1}{w}\right| dw.$$

   Observe that $w > 0$ and $0 < \frac{z}{w} < 1$ so

   $$\begin{aligned} f_Z(z) &= \int_z^{\infty} \frac{1}{2} w^2 e^{-w} \cdot 2\frac{z}{w} \cdot \frac{1}{w} dw \\ &= ze^{-z}u(z). \end{aligned}$$

b. Let X be uniformly distributed over the interval $(\theta, \theta+1)$. A nontrivial function $h(x)$ such that $E[h(X)] = 0$ is called an *unbiased estimator of zero* for this pdf (nontrivial here means $h(x)$ is not simply 0). This kind of function has applications in estimation theory (when θ is unknown). Note $h(x)$ does not depend on θ.

i. For this pdf does $h(x)$ have to be periodic? If so, prove it and find the maximum permitted period. If $h(x)$ does not have to be periodic, explain why not.

Solution: Yes, $h(x)$ must be periodic. To see this note that

$$E[h(X)] = \int_{\theta}^{\theta+1} h(x)dx = 0 \text{ for all } \theta$$

thus

$$0 = \frac{d}{d\theta}\int_{\theta}^{\theta+1} h(x)dx = h(\theta+1) - h(\theta) \text{ for all } \theta.$$

Hence, $h(x)$ must be periodic with a maximum period of 1.

ii. Give an example of such an $h(x)$ as described above.

Solution: Let $h(x) = \sin(2\pi x)$.