EE 503

Quiz 7 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (8 points.) Consider the joint density for random variables X and Y

$$f(x,y) = \begin{cases} (x+2y)/4, & 0 < x < 2,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the marginal density of X, $f_X(x)$.

Solution: We find

$$f_X(x) = \int_0^1 \frac{1}{4}(x+2y)\,dy$$

which becomes

$$f_X(x) = \begin{cases} \frac{1}{4}(x+1), & 0 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find the pdf of the random variable $Z = 1/(X+1)^2$.

Solution: Note that the function $g(x) = 1/(x+1)^2$ is monotone over the domain $0 < x < 2$ and is differentiable so we have by the theorem given in class

$$f_Z(z) = f_X\left(g^{-1}(z)\right)\left|\frac{dg^{-1}(z)}{dz}\right|.$$

Here $z = g(x)$ so

$$x = g^{-1}(z) = \frac{1}{\sqrt{z}} - 1$$

and

$$\frac{dx}{dz} = -\frac{1}{2z^{3/2}}$$

so

$$f_Z(z) = \begin{cases} \dfrac{1}{8z^2}, & 1 < z < 9 \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 2. (7 points.) An exponential random variable (just call it X) has pdf

$$f(x) = \begin{cases} \frac{1}{\beta}e^{-x/\beta}, & 0 \leq x < \infty, \quad \beta > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

This is denoted $X \sim Exponential(\beta)$. Now for our case let $X \sim Exponential(\beta)$ and let $Y \sim Exponential(\mu)$ and suppose X and Y are independent random variables. Let $Z = \max\{X, Y\}$.

a. Find the pdf of Z.

Solution: We find

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(\max\{X,Y\} \leq z) = P(X \leq z, Y \leq z) \\ &= P(X \leq z)P(Y \leq z) = F_X(z)F_Y(z) = \left(1 - e^{-z/\beta}\right)\left(1 - e^{-z/\mu}\right). \end{aligned}$$

Then

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \left(1 - e^{-z/\beta}\right)\left(\frac{1}{\mu}e^{-z/\mu}\right) + \left(1 - e^{-z/\mu}\right)\left(\frac{1}{\beta}e^{-z/\beta}\right)$$

b. Compute $P\left(X - 1 < Y < X)\right)$.

Solution: The joint density of X and Y is

$$f(x,y) = \begin{cases} \dfrac{1}{\beta\mu}e^{-x/\beta}e^{-y/\mu}, & 0 \leq x, y < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\begin{aligned} P\left(X - 1 < Y < X)\right) &= \int_0^1 \int_0^x f(x,y)dydx + \int_1^\infty \int_{x-1}^x f(x,y)dydx \\ &= \frac{\beta(1 - e^{-1/\beta})}{\beta + \mu}. \end{aligned}$$