EE 503

Quiz 6 Solution

Fall 2019, 15 Minutes, 15 Points

a. Suppose X is a random variable with density $f(x) = 2x$ for $0 \leq x \leq 1$ and is 0 elsewhere. Find the variance of X.

Solution: We find

$$E[X] = \int_0^1 xf(x)dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

$$E\left[X^2\right] = \int_0^1 x^2 f(x)dx = \int_0^1 2x^3 dx = \frac{1}{2}$$

Thus, $Var(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} = 0.0556$.

b. Suppose Y is a random variable with moment generating function $M_Y(s) = \dfrac{1}{1-\beta s}$ for $0 \leq s < \dfrac{1}{\beta}$, $\beta > 0$. Find the variance of Y.

Solution: We find

$$E[Y] = M_Y'(s)|_{s=0} = \beta$$

$$E\left[Y^2\right] = M_Y''(s)|_{s=0} = 2\beta^2$$

Thus, $Var(Y) = 2\beta^2 - \beta^2 = \beta^2$.

Problem 2. (7 points.) Probability bounds.

a. Suppose X is a random variable with finite mean μ and finite variance σ^2. Using the Chebyshev inequality find a bound for $P(|X - \mu| \geq k\sigma)$ where k is a positive real number.

Solution: We find

$$P(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}.$$

b. Suppose Z is a standard normal (Gaussian) random variable. Show

$$P(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \cdot \frac{e^{-t^2/2}}{t}, \text{ for all } t > 0.$$

Solution: We find

$$\begin{aligned} P(|Z| \geq t) &= 2P(Z \geq t) \\ &= 2\int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &\leq \sqrt{\frac{2}{\pi}} \int_t^\infty \frac{z}{t} e^{-z^2/2} dz \quad \text{(since } z/t > 1 \text{ for } z > t) \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{e^{-t^2/2}}{t} \quad \text{(using integration by parts)} \end{aligned}$$