EE 503

Quiz 5 Solution

Fall 2019, 15 Minutes, 15 Points

Problem 1. (8 points.) The continuous random variable X has pdf

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find $E[X]$.

   Solution:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{0}^{\infty} 2xe^{-2x} dx \\ &= 1/2 \end{aligned}$$

   using integration by parts.

b. This random variable has cdf $F(x) = 1 - e^{-2x}$ for $x > 0$ and is 0 elsewhere. Suppose the random variable U is uniform in $[0, 1)$. Find a function $g(u)$ such that the random variable $g(U)$ has the same cdf as X.

   Solution: As demonstrated in class we solve

$$u = F(x) \Rightarrow u = 1 - e^{-2x} \Rightarrow x = -\frac{1}{2} \ln(1 - u)$$

   so $g(u) = -\frac{1}{2} \ln(1 - u)$.

Note: Although not needed here you could also use $g(u) = -\frac{1}{2} \ln(u)$ in practice since $1 - U$ is uniform in $(0, 1]$ which is the same as U except for the endpoints (which do not matter since they have probability 0 of occurring).

Problem 2. (7 points.) Recall that the geometric random variable X can be thought of as the trial at which the first success occurs in a sequence of independent Bernoulli trials with probability of success p on each trial. We got $P(X = k) = (1 - p)^{k-1}p$. Suppose instead we let X denote the trial at which the rth success occurs. Derive a formula for $P(X = k)$ in this case. *Hint:* Think about the binomial distribution.

Solution: The event $\{X = k\}$ means there were exactly $r - 1$ successes in the first $k - 1$ trials and a successes on the kth trial. Thus,

$$
\begin{aligned}
P(X = k) &= \binom{k-1}{r-1} p^{r-1}(1-p)^{k-1-(r-1)}p \\
&= \binom{k-1}{r-1} p^{r}(1-p)^{k-r}.
\end{aligned}
$$

This random variable has a *negative binomial* distribution.