EE 503

Homework 13

Due Wednesday December 4, 2019 at 6 p.m.

Work all 10 problems.

Problem 1. Suppose you are going to conduct a political poll. You wish to have a margin of error of $\pm$ 3% and a 95% confidence interval. Determine the number of people you need to poll.

Problem 2. Suppose you are going to perform a bit error rate (BER) simulation. How many bit errors should you count so that you are 95% confident the true BER is within $\pm$ 5% of the BER calculated from you simulation?

Problem 3. Suppose you perform a simulation of a communication system and you are processing received bits. When errors occur in these bits the errors are independent. Suppose after processing 10 million bits you observe no errors. How confident are you that the true bit error rate is no higher than 10^{-5}?

Problem 4. Suppose you gather some data and compute the sample mean and it has a value of 7.7 based on a sample size of 200. Because of the relatively large sample size we can assume the sample mean is normal (or Gaussian).

a. Construct a 95% confidence interval for the true mean of the data if the true variance of the data is known to be 1.2.

b. Construct a 95% confidence interval for the true mean of the data if the variance of the data is unknown but the sample variance is 1.2.

Problem 5. Let $X = (X_1, X_2, \ldots, X_n)$ be i.i.d. where each $X_i \sim U(\theta_1, \theta_2)$ where θ_1 and θ_2 are unknown. Find the MLE for θ_1 and θ_2.

Problem 6. Let $X = (X_1, X_2, \ldots, X_n)$ be i.i.d. where each $X_i \sim U(\theta, 2\theta)$ where θ is unknown. Find the MLE for θ.

Problem 7. Let $X = (X_1, X_2, \dots, X_n)$ be i.i.d. where each $X_i \sim U(a - \theta, a + \theta)$ where both a and θ are unknown. Find the MLE for a and θ.

Problem 8. Let $X = (X_1, X_2, \dots, X_n)$ be i.i.d. where each X_i is Poisson distributed with parameter $\lambda \geq 0$, that is, each X_i has probability mass function

$$P(X_i = x_i) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}, \quad i = 0, 1, 2, \dots.$$

Find the MLE for λ.

Problem 9. Let $X = (X_1, X_2, \dots, X_n)$ be i.i.d. where each X_i is Poisson distributed with parameter $\lambda \geq 0$, that is, each X_i has probability mass function

$$P(X_i = x_i) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}, \quad i = 0, 1, 2, \dots.$$

Find the Cramer-Rao Lower Bound for an unbiased estimator of λ. Note that the Poisson distribution is a member of the exponential family. Does the MLE for λ found in Problem 8 achieve this lower bound?

Problem 10. Let $X = (X_1, X_2, \dots, X_n)$ be i.i.d. where each X_i has pdf $f(x|\theta) = \frac{1}{\theta}$, $0 < x < \theta$, and zero elsewhere.

a. Show the Cramer-Rao Lower Bound for the variance of any estimator, W, of θ satisfies

$$Var_\theta W \geq \frac{\theta^2}{n}.$$

b. Let $Y = X_{(n)}$, that is, Y is the maximum value observed. Let

$$Z = \frac{n+1}{n} Y.$$

Show Z is an unbiased estimator of θ.

c. Show

$$Var_\theta Z = \frac{\theta^2}{n(n+2)}.$$

d. The variance of Z is smaller than the Cramer-Rao Lower Bound. Explain, mathematically, why this does not violate the Cramer-Rao Inequality. Note: for this part you may need to use

Leibnitz's Rule: If $f(x, \theta), a(\theta)$ and $b(\theta)$ are differentiable with respect to θ then

$$\begin{aligned}\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx &= f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) \\ &+ \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\theta} f(x,\theta)dx.\end{aligned}$$