EE 503

Homework 12 Solution

Instructor: Christopher Wayne Walker, TA: James Huang

Problem 1. Let S be a set and n a natural number. Recall

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k.$$

Show

$$\limsup_{n\to\infty} A_n = \{x \in S : x \in A_n \text{ for infinitely many } n\}.$$

Solution:
We want to show the two sets are identical:

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{x \in S : x \in A_n \text{ for infinitely many } n\}.$$

Thus, we pick an arbitrary $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. This implies that $\forall n \in \mathbb{N}$, $x \in \bigcup_{k=n}^{\infty} A_k$ which tells us that $\forall n \in \mathbb{N}, \exists k \geq n$, s.t. $x \in A_k$, or equivalently, $x \in A_n$ for infinitely many n.
Conversely, we pick an arbitrary $x \in A_n$ for infinitely many n, then $\forall n \in \mathbb{N}, \exists k \geq n$, s.t. $x \in A_k$, or equivalently, $\forall n \in \mathbb{N}, x \in \bigcup_{k=n}^{\infty} A_k$. Thus, $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.
Since we show that $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \subseteq \{x \in S : x \in A_n \text{ for infinitely many } n\}$ and $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \supseteq \{x \in S : x \in A_n \text{ for infinitely many } n\}$, we get

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{x \in S : x \in A_n \text{ for infinitely many } n\}.$$

Problem 2. Let X_k, $k = 1, 2, \ldots$ be a sequence of independent normal (Gaussian) random variables with mean 0 and variance σ^2. Define

$$Y_n = \frac{1}{N} \sum_{k=n-N+1}^{n} X_k, \quad n = 0, 1 \ldots$$

where we take $Y_n = 0$ for $n < 0$.

This kind of operation of averaging is often used in signal processing. It is a type of low pass filter (LPF) since if the X_k had sinusoidal components they would average out (that is, be filtered out). We will refer to this type of filter as an I&D (integrate and dump) filter.

Now for implementation purposes we can also construct a LPF to approximate the above averaging filter using an IIR (infinite impulse response) filter as

$$\tilde{Y}_n = (1-\alpha)X_n + \alpha\tilde{Y}_{n-1}, \quad n = 0, 1 \dots$$

where we take $\tilde{Y}_{-1} = 0$ and $\alpha \in (0,1)$.

a. Show that $E[\tilde{Y}_n] = E[Y_n] = 0$ for any $\alpha \in (0,1)$.

b. Determine the value of $\alpha = \alpha(N)$ so that the variance of the IIR filter output ($\tilde{Y}_n$) matches the variance of the I&D filter output (Y_n) as $n \to \infty$.

Solution:

a.

$$\begin{aligned}
\mathbf{E}[Y_n] &= \mathbf{E}\left[\frac{1}{N}\sum_{k=n-N+1}^{n} X_k\right] = 0, \\
\mathbf{E}[\tilde{Y}_n] &= \mathbf{E}\left[(1-\alpha)X_n + \alpha(1-\alpha)X_{n-1} + \alpha^2(1-\alpha)X_{n-2} + \cdots + \alpha^n(1-\alpha)X_0 + \alpha^{n+1}\tilde{Y}_{-1}\right] \\
&= 0.
\end{aligned}$$

b.

$$\begin{aligned}
\mathrm{var}(Y_n) &= \mathrm{var}\left(\frac{1}{N}\sum_{k=n-N+1}^{n} X_k\right) = \frac{1}{N^2}\cdot N\sigma^2 = \frac{\sigma^2}{N}, \\
\mathrm{var}(\tilde{Y}_n) &= \sigma^2(1-\alpha)^2\left(1+\alpha^2+\alpha^4+\cdots+\alpha^{2n}\right) \\
&= \sigma^2(1-\alpha)^2\cdot\frac{1-\alpha^{2(n+1)}}{1-\alpha^2} \\
&= \sigma^2\cdot\frac{(1-\alpha)\left(1-\alpha^{2(n+1)}\right)}{1+\alpha} \to \sigma^2\cdot\frac{1-\alpha}{1+\alpha} \text{ as } n\to\infty \text{ if } \alpha\in(0,1).
\end{aligned}$$

Hence, $\frac{1}{N} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = \frac{N-1}{N+1}$.