EE 503

Homework 11 Solution

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Problem 1. Suppose ω is selected at random in the interval [0, 1]. For each of the following state whether the sequence of random variables converges surely, almost surely or not at all. If the sequence does converge indicate the random variable or constant to which the sequence converges.

a. $X_n(\omega) = \dfrac{\omega}{n}$.

b. $Y_n(\omega) = \omega\left(1 - \dfrac{1}{n}\right)$.

c. $Z_n(\omega) = \omega e^n$.

d. $V_n(\omega) = \omega^n$.

e. $W_n(\omega) = \cos^n 2\pi\omega$.

Solution:

a. $X_n \xrightarrow{e} 0$, since

$$\forall \omega \in [0,1], \lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} \frac{\omega}{n} = 0.$$

b. $Y_n \xrightarrow{e} U \sim \text{Uniform}([0,1])$, since

$$\forall \omega \in [0,1], \lim_{n\to\infty} Y_n(\omega) = \lim_{n\to\infty} \omega\left(1 - \frac{1}{n}\right) = \omega.$$

c. The sequence of random variables Z_n does not converge.

d. $V_n \xrightarrow{a.s.} 0$, since

$$P\left[\left\{\omega \in [0,1] \middle| \lim_{n\to\infty} V_n(\omega) = 0\right\}\right] = P[\{\omega \in [0,1)\}] = 1.$$

e. $W_n \xrightarrow{a.s.} 0$, since

$$P\left[\left\{\omega \in [0,1] \middle| \lim_{n\to\infty} W_n(\omega) = 0\right\}\right] = P[\{\omega \in (0,1/2) \cup (1/2,1)\}] = 1.$$

Problem 2. Let X_n be a sequence of i.i.d. equiprobable Bernoulli random variables and let

$$Y_n = 2^n X_1 X_2 \dots X_n.$$

a. Show this sequence converges almost surely and indicate the limit.

b. Determine whether or not Y_n converges in the mean square sense.

Solution:

a. $Y_n \stackrel{a.s.}{\to} 0$ since

$$\begin{aligned} P[\lim_{n\to\infty} Y_n = 0] &= P[\{\text{An infinite sequence of i.i.d. Bernoulli}(1/2) \text{ r.v.s has a } 0\}] \\ &= 1. \end{aligned}$$

b. Y_n converges to 0 in mean square sense if $\mathbf{E}[(Y_n - 0)^2] \to 0$ as $n \to \infty$. However,

$$\begin{aligned} \mathbf{E}[Y_n^2] &= 2^{2n} \cdot \left(\frac{1}{2}\right)^n + 0 \cdot \left(1 - \left(\frac{1}{2}\right)^n\right) \\ &= 2^n \end{aligned}$$

does not converge.

Problem 3. A random variable X is said to be Laplacian with parameter $\alpha > 0$ if it has density

$$f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}, \quad -\infty < x < \infty.$$

Let X_n be a sequence of Laplacian random variables with parameter $\alpha = n$. Show this sequence converges in probability (and hence in distribution).

Solution:

Let $X_n \sim \text{Laplacian}(n)$, $X_n \xrightarrow{p} 0$, since for an arbitrary $\epsilon > 0$

$$\begin{aligned} P[|X_n - 0| \leq \epsilon] &= P[|X_n| \leq \epsilon] \\ &= \int_{-\epsilon}^{\epsilon} f_{X_n}(x)dx \\ &= 2\int_0^{\epsilon} \frac{n}{2} e^{-nx} dx \\ &= 1 - e^{-n\epsilon} \to 1 \text{ as } n \to \infty. \end{aligned}$$

Problem 4. We know that convergence in probability always implies convergence in distribution but the converse is not, in general, true. However, suppose the random sequence Z_n converges to Z in distribution where Z is some constant z_0. In this case show that $Z_n \to Z$ in distribution implies $Z_n \to Z$ in probability.

Solution:
Since Z_n converges to a constant random variable $Z = z_0$ in distribution, we know

$$F_{Z_n}(z) \to F_Z(z) = u(z - z_0), \forall z \in \mathbb{R} \text{ except for } z = z_0,$$

where $u(\cdot)$ denotes the unit step function. To show that Z_n converges in probability as well, we analyze $P[|Z_n - z| \leq \epsilon]$ (or $P[|Z_n - z| > \epsilon]$) for an arbitrary ϵ,

$$\begin{aligned} P[|Z_n - z| \leq \epsilon] &= P[z - \epsilon \leq Z \leq z + \epsilon] \\ &= F_{Z_n}(z + \epsilon) - F_{Z_n}(z - \epsilon) \\ &\to F_Z(z + \epsilon) - F_Z(z - \epsilon) \text{ as } n \to \infty \\ &= 1. \end{aligned}$$

Since $\forall \epsilon > 0$, $\lim_{n\to\infty} P[|Z_n - z| > \epsilon] = 0$, we have $Z_n \xrightarrow{p} z$.

Problem 5. A particular Cauchy random variable X has density

$$f(x) = \frac{1}{\pi}\frac{1}{1 + x^2}, \quad -\infty < x < \infty.$$

X does not have a mgf since its moments do not exist. Show, in fact, that $E[X]$ does not exist by trying to analytically compute the mean of X.

Solution:
Cauchy random variable centered at 0 does not have a mean since the following doubly limit integral is not properly defined

$$\begin{aligned}\lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b \frac{1}{\pi}\frac{x}{1+x^2}\,dx &= \lim_{a\to-\infty}\lim_{b\to\infty}\frac{1}{2\pi}\ln(1+x^2)\Big|_{x=a}^{b} \\ &= \frac{1}{2\pi}\left[\lim_{b\to\infty}\ln(1+b^2) - \lim_{a\to-\infty}\ln(1+a^2)\right] \\ &= \frac{1}{2\pi}[\infty-\infty] = ?\end{aligned}$$

The reason we cannot subtract infinity with infinity is because they can be approaching in different speed e.g. considering taking the limit with $a = b+1$, $a = 2b$ or $a = b^2$.
Still, there is one thing that is going to give you a zero. It is the principle value integral.

$$\begin{aligned}⨍_{-\infty}^{\infty} \frac{1}{\pi}\frac{x}{1+x^2}\,dx &= \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{\pi}\frac{x}{1+x^2}\,dx \\ &= \lim_{a\to\infty}\frac{1}{2\pi}\ln(1+x^2)\Big|_{x=-a}^{a} \\ &= 0.\end{aligned}$$

However, this cannot be used to say that the mean of a Cauchy random variable is 0. That is, the mean is defined as the value of the integral in the usual sense and not in the principal value sense. In conclusion, without specifying how the two infinities were approached, the mean is not properly defined.