EE 503

Homework 10 Solution

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Problem 1. Suppose the random variables $X_i,\ i = 1, 2, ..., n$ are uncorrelated and have the same mean μ and variance σ^2. Define the sample mean $\bar{X}$ as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the sample variance $\bar{V}$ as

$$\bar{V} = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2.$$

Show

a. $E\left(\bar{X}\right) = \mu$.

b. $Var\left(\bar{X}\right) = \sigma^2/n$.

c. $E\left(\bar{V}\right) = \sigma^2$.

Solution:

a. Since $\{X_i\}_{i=1}^{n}$ is a sequence of iid random variables with mean $\mathbf{E}[X_i] = \mu$ and variance $\text{var}(X_i) = \sigma^2$, $\forall i \in \{1, 2, \cdots, n\}$, the expectation of the sample mean $\bar{X}$ is

$$\mathbf{E}[\bar{X}] = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[X_i] = \frac{1}{n}\cdot n\mu = \mu,$$

b. while the variance of the sample mean $\bar{X}$ is

$$
\begin{aligned}
\mathrm{var}(\bar{X}) &= \mathbf{E}\left[(\bar{X}-\mu)^2\right] \\
&= \mathbf{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\right)^2\right] \\
&= \frac{1}{n^2}\mathbf{E}\left[\left(\sum_{i=1}^{n}(X_i-\mu)\right)^2\right] \\
&= \frac{1}{n^2}\left(n\sigma^2 + n(n-1)\cdot 0\right), \text{only } n \text{ out of } n^2 \text{ terms are nonzero} \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

Note that $n(n-1)$ terms in the expectation are zeros due to independence.

c. The expectation of the sample variance $\bar{V}$ is

$$
\begin{aligned}
\mathbf{E}[\bar{V}] &= \mathbf{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2\right] \\
&= \frac{1}{n-1}\mathbf{E}\left[\sum_{i=1}^{n}((X_i-\mu)-(\bar{X}-\mu))^2\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\mathbf{E}\left[(X_i-\mu)^2 - 2(X_i-\mu)(\bar{X}-\mu) + (\bar{X}-\mu)^2\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\sigma^2 - 2\mathbf{E}\left[(X_i-\mu)(\bar{X}-\mu)\right] + \frac{\sigma^2}{n}\right).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\mathbf{E}[(X_i-\mu)(\bar{X}-\mu)] &= \mathbf{E}\left[(X_i-\mu)\frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)\right] \\
&= \frac{1}{n}\sigma^2,
\end{aligned}
$$

since $\mathbf{E}\left[(X_i-\mu)\frac{1}{n}\sum_{j=1,j\neq i}^{n}(X_j-\mu)\right]=0$. Finally,

$$\mathbf{E}[\bar{V}]=\frac{1}{n-1}\sum_{k=1}^{n}\left(\sigma^2-\frac{2\sigma^2}{n}+\frac{\sigma^2}{n}\right)=\sigma^2.$$

Problem 2. Suppose the random variable X is normally distributed with mean 2 and variance 9. Find

a. $P(X<5)$.

b. $P(X>-1)$.

c. $P(-1<X<5)$.

d. $P(X<10)$.

Solution:
Let $\Phi(z)$ be the standard normal cdf, i.e. $Z\sim\mathcal{N}(0,1)$ and

$$\Phi(z)=F_Z(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt.$$

a. $P[X<5]=\Phi\left(\frac{5-2}{3}\right)=\Phi(1)=0.8413$.

b. $P[X>-1]=1-\Phi\left(\frac{-1-2}{3}\right)=1-\Phi(-1)=0.8413$.

c. $P[-1<X<5]=\Phi(1)-\Phi(-1)=0.6827$.

d. $P[X<10]=\Phi\left(\frac{10-2}{3}\right)=0.9962$.

Problem 3. Let X be binomially distributed with $n=50$ and $p=\frac{1}{3}$. Find

a. $P(X=17)$.

b. $P(X\leq 20)$.

Solution:

a. $P[X=17]=\binom{50}{17}p^{17}(1-p)^{33}=0.1178$.

b. $P[X \leq 20] = \sum_{k=0}^{20} \binom{50}{k} p^k (1-p)^{50-k} = 0.8741.$

Problem 4. We can think of X in Problem 3 as the sum of 50 Bernoulli random variables with success probability $p = \frac{1}{3}$. So let us use the CLT to approximate our answers in Problem 3. Using the CLT find

a. $P(X = 17)$.

b. $P(X \leq 20)$.

Remember to use continuity correction as we discussed in our class example.

Solution:
Using CLT with $\frac{S-\mu_S}{\sigma} \sim \mathcal{N}(0,1)$, $\mu_S = np = \frac{50}{3}$ and $\sigma_S = \sqrt{np(1-p)} = \frac{10}{3}$, we have

a. $P[X = 17] \approx P\left[\frac{16.5-\mu_S}{\sigma_S} \leq Z \leq \frac{17.5-\mu_S}{\sigma_S}\right] = 0.1186,$

b. $P[X \leq 20] \approx P\left[Z \leq \frac{20.5-\mu_S}{\sigma_S}\right] = 0.8749.$