EE 503

Homework 9 Solution

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Problem 1. Let $X \sim N\left(\mu_1, \sigma_1^2\right)$. Find a, b such that if $Y = aX + b$ then $Y \sim N\left(\mu_2, \sigma_2^2\right)$.

Solution:
Since $Y = aX + b$ is a linear function of Gaussian random variable X, Y is also a Gaussian random variable. Then, we can find its mean and variance,

$$\mathbf{E}[Y] = \mathbf{E}[aX + b] = a\mu_1 + b,$$
$$\mathrm{var}(Y) = \mathrm{var}(aX + b) = a^2\sigma_1^2,$$

Therefore,

$$a^2\sigma_1^2 = \sigma_2^2 \Rightarrow a = \pm\frac{\sigma_2}{\sigma_1},$$
$$a\mu_1 + b = \mu_2 \Rightarrow b = \mu_2 \mp \frac{\sigma_2}{\sigma_1}\mu_1.$$

Problem 2. Let X and Y have joint *pdf*

$$f_{XY}(x, y) = \begin{cases} 2e^{-(x+y)}, & 0 \leq y \leq x < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Z = X + Y$. Show Z has *pdf*

$$f_Z(z) = ze^{-z}u(z).$$

Note: X and Y are not independent.

Solution:

$$\begin{aligned}
F_Z(z) &= P[Z \leq z] \\
&= P[X + Y \leq z] \\
&= \int\int_{A_z} f_{X,Y}(x, y)\, dxdy, \text{ where } A_z \text{ is the triangle with vertices } (0,0), (z,0), (z/2, z/2). \\
&= \int_0^{\frac{z}{2}} \int_y^{z-y} 2e^{-x}e^{-y}\, dxdy, \text{ if } z \geq 0 \\
&= 1 - e^{-z} - ze^{-z}, \text{ if } z \geq 0.
\end{aligned}$$

Hence,

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= z e^{-z}, \text{ if } z \geq 0. \end{aligned}$$

Problem 3. Consider the random variable X with the Pareto density

$$f(x) = \begin{cases} \lambda x^{-\lambda-1}, & x > 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = \ln(X)$ (the natural log). Find the density function for Y.

Solution:

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[\ln(X) \leq y] \\ &= P[X \leq e^y] \\ &= F_X(e^y). \end{aligned}$$

Hence,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= f_X(e^y) \cdot e^y \\ &= \lambda e^{-(\lambda+1)y} \cdot e^y, \text{ if } e^y > 1 \\ &= \lambda e^{-\lambda y}, \text{ if } y > 0. \end{aligned}$$

Problem 4. Let Z_1 and Z_2 be independent standard normal random variables. Let $Y_1 = Z_1 + Z_2$ and let $Y_2 = Z_2 - Z_1$.

a. Show that Y_1 and Y_2 are independent. *Hint*: You can compute the joint characteristic function (or the joint moment generating function) for Y_1 and Y_2 and show that it factors (no integration is necessary).

b. Find the joint density function for Y_1 and Y_2. *Hint*: Make use of independence.

Solution:

a. Recall that Y_1 and Y_2 are independent if and only if $\Phi_{Y_1,Y_2}(\omega_1,\omega_2) = \Phi_{Y_1}(\omega_1)\Phi_{Y_2}(\omega_2), \forall \omega_1,\omega_2 \in \mathbb{R}$ and $X \sim \mathcal{N}(\mu,\sigma^2) \Rightarrow \Phi_X(\omega) = e^{j\mu\omega - \frac{1}{2}\sigma^2\omega^2}$.

$$\begin{aligned}
\Phi_{Y_1,Y_2}(\omega_1,\omega_2) &= \mathbf{E}\left[e^{j(\omega_1 Y_1 + \omega_2 Y_2)}\right] \\
&= \mathbf{E}\left[e^{j(\omega_1 Z_1 + \omega_1 Z_2 - \omega_2 Z_1 + \omega_2 Z_2)}\right] \\
&= \Phi_{Z_1,Z_2}(\omega_1 - \omega_2, \omega_1 + \omega_2) \\
&= \Phi_{Z_1}(\omega_1 - \omega_2)\Phi_{Z_2}(\omega_1 + \omega_2), \text{ since } Z_1 \perp\!\!\!\perp Z_2 \\
&= e^{-\frac{1}{2}(\omega_1-\omega_2)^2} e^{-\frac{1}{2}(\omega_1+\omega_2)^2}. \\
\Phi_{Y_1}(\omega_1)\Phi_{Y_2}(\omega_2) &= \mathbf{E}[e^{j\omega_1 Y_1}]\mathbf{E}[e^{j\omega_2 Y_2}] \\
&= \mathbf{E}[e^{j\omega_1(Z_1+Z_2)}]\mathbf{E}[e^{j\omega_2(-Z_1+Z_2)}] \\
&= \mathbf{E}[e^{j\omega_1 Z_1}]\mathbf{E}[e^{j\omega_1 Z_2}]\mathbf{E}[e^{-j\omega_2 Z_1}]\mathbf{E}[e^{j\omega_2 Z_2}], \text{ since } Z_1 \perp\!\!\!\perp Z_2 \\
&= e^{-\frac{1}{2}\omega_1^2} \cdot e^{-\frac{1}{2}\omega_1^2} \cdot e^{-\frac{1}{2}\omega_2^2} \cdot e^{-\frac{1}{2}\omega_2^2}.
\end{aligned}$$

Since $\Phi_{Y_1,Y_2}(\omega_1,\omega_2) = \Phi_{Y_1}(\omega_1)\Phi_{Y_2}(\omega_2)$, Y_1 and Y_2 are independent.

b. Since Y_1 and Y_2 are independent and $Y_1, Y_2 \sim \mathcal{N}(0,2)$, their joint pdf is

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi \cdot 2} e^{-\left(\frac{y_1^2}{4} + \frac{y_2^2}{4}\right)}, \forall y_1, y_2 \in \mathbb{R}.$$

Problem 5. Let Z_1 and Z_2 be independent random variables each having an exponential density of the form $f_Z(z) = \lambda e^{-\lambda z} U(z)$. Define $X = Z_2$, $Y = Z_2(1 + Z_1)$. Find

a. Find $E(Y|X = x)$.

b. Find $E\left(E(Y|X)\right)$.

c. Find $Var\left(E(Y|X)\right)$.

d. Find $Var\left(Y|X = x\right)$.

e. Find $E\left(Var(Y|X)\right)$.

f. Find the best MSE predictor of Y given $X = x$.

g. Find the best linear MSE predictor of Y based on X.

Solution:

a.

$$\begin{aligned}
\mathbf{E}[Y|X=x] &= \mathbf{E}[X(1+Z_1)|X=x] \\
&= x \cdot \mathbf{E}[1+Z_1|X=x] \\
&= x \cdot \mathbf{E}[1+Z_1], \text{ since } Z_1 \text{ and } X = Z_2 \text{ are independent} \\
&= x\left(1+\frac{1}{\lambda}\right).
\end{aligned}$$

b. Notice that $\mathbf{E}[Y|X]$ is a function of X,

$$\begin{aligned}
\mathbf{E}[\mathbf{E}[Y|X]] &= \int \mathbf{E}[Y|X=x] f_X(x)dx \\
&= \left(1+\frac{1}{\lambda}\right)\int x f_X(x)dx \\
&= \left(1+\frac{1}{\lambda}\right)\frac{1}{\lambda} \\
&= \frac{1}{\lambda}+\frac{1}{\lambda^2}.
\end{aligned}$$

c.

$$\begin{aligned}
\text{var}(\mathbf{E}[Y|X]) &= \text{var}\left(X\left(1+\frac{1}{\lambda}\right)\right) \\
&= \left(1+\frac{1}{\lambda}\right)^2 \text{var}(X) \\
&= \left(1+\frac{1}{\lambda}\right)^2 \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}+\frac{2}{\lambda^3}+\frac{1}{\lambda^4}.
\end{aligned}$$

d.

$$\begin{aligned}
\text{var}\,(Y|X=x) &= \mathbf{E}[Y^2|X=x] - (\mathbf{E}[Y|X=x])^2 \\
&= x^2\mathbf{E}[(1+Z_1)^2] - x^2\left(1+\frac{1}{\lambda}\right)^2 \\
&= x^2\mathbf{E}[1+2Z_1+Z_1^2] - x^2\left(1+\frac{1}{\lambda}\right)^2 \\
&= x^2\left(1+\frac{2}{\lambda}+\frac{2}{\lambda^2}-1-\frac{2}{\lambda}-\frac{1}{\lambda^2}\right) \\
&= \frac{x^2}{\lambda^2}.
\end{aligned}$$

e.

$$\begin{aligned}
\mathbf{E}[\text{var}(Y|X)] &= \int \frac{x^2}{\lambda^2} f_X(x)\,dx \\
&= \frac{1}{\lambda^2}\mathbf{E}[X^2] \\
&= \frac{1}{\lambda^2}\cdot\frac{2}{\lambda^2} \\
&= \frac{2}{\lambda^4}.
\end{aligned}$$

f.

$$\begin{aligned}
\hat{Y}_{MMSE} &= \mathbf{E}[Y|X=x] \\
&= x\left(1+\frac{1}{\lambda}\right).
\end{aligned}$$

g.

$$
\begin{aligned}
\operatorname{cov}(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[X^2(1+Z_1)] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[X^2]\mathbf{E}[1+Z_1] - \frac{1}{\lambda}\left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right) \\
&= \frac{2}{\lambda^2}\left(1 + \frac{1}{\lambda}\right) - \frac{1}{\lambda}\left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right) \\
&= \frac{1}{\lambda^2} + \frac{1}{\lambda^3}. \\
\hat{Y}_{LMMSE} &= \mathbf{E}[Y] + \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}\left(x - \mathbf{E}[X]\right) \\
&= \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right) + \frac{\frac{1}{\lambda^2} + \frac{1}{\lambda^3}}{\frac{1}{\lambda^2}}\left(x - \frac{1}{\lambda}\right) \\
&= \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right) + \left(1 + \frac{1}{\lambda}\right)\left(x - \frac{1}{\lambda}\right) \\
&= x\left(1 + \frac{1}{\lambda}\right).
\end{aligned}
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