EE 503

Homework 8 Solution

Instructor: Christopher Wayne Walker, TA: James Huang

Problem 1. Suppose X and Y are independent random variables, each normally distributed with mean 0 and variance σ^2.

Compute $E[|X-Y|]$.

Solution:
Since X and $-Y$ are two independent $\mathcal{N}(0,\sigma^2)$ random variables, $Z = X + (-Y) \sim \mathcal{N}(0, 2\sigma^2)$. We can thus find the expected value as

$$\begin{aligned}
\mathbf{E}[|X-Y|] &= \mathbf{E}[|X|] \\
&= \int_{-\infty}^{\infty} |z| f_Z(z) dz \\
&= \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi \cdot 2\sigma^2}} e^{-\frac{z^2}{2\cdot 2\sigma^2}} dz \\
&= 2 \cdot \frac{1}{\sqrt{2\pi \cdot 2\sigma^2}} \int_{0}^{\infty} z e^{-\frac{z^2}{4\sigma^2}} dz \\
&= \frac{1}{\sqrt{\pi}\sigma} \left(-2\sigma^2 e^{-\frac{z^2}{4\sigma^2}} \right)_{z=0}^{\infty} \\
&= \frac{2\sigma}{\sqrt{\pi}}.
\end{aligned}$$

Problem 2. The two-dimensional continuous random variable (X, Y) has joint *pdf*

$$f_{XY}(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 \le x \le 1,\ 0 \le y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Compute $f_{X|Y}(x|y)$.

b. Find $E[X|Y=y]$.

Solution:

a.

$$
\begin{aligned}
f_Y(y) &= \int_0^1 x^2 + \frac{1}{3}xy\,dx \\
&= \frac{1}{3}x^3 + \frac{1}{6}x^2y\Big|_{x=0}^{1} \\
&= \frac{1}{3} + \frac{1}{6}y. \\
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{6x^2 + 2xy}{2+y}, \text{ if } x \in [0,1] \text{ and } y \in [0,2].
\end{aligned}
$$

b.

$$
\begin{aligned}
\mathbf{E}[X|Y=y] &= \int x f_{X|Y}(x|y)dx \\
&= \int_0^1 \frac{6x^3 + 2x^2y}{2+y}dx \\
&= \frac{\frac{3}{2}x^4 + \frac{2}{3}x^3y}{2+y}\Big|_{x=0}^{1} \\
&= \frac{9+4y}{2+y}.
\end{aligned}
$$

Problem 3. Suppose the two-dimensional continuous random variable (X, Y) has joint *pdf*

$$
f_{XY}(x,y) = \begin{cases} x+y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}
$$

a. Find $f_{X|Y}(x|y)$.

b. Let $B = \{X + Y \geq 1/4\}$. Find $P(B)$.

Solution:

a.

$$\begin{aligned}
f_Y(y) &= \int_0^1 x + y\,dx \\
&= \frac{1}{2}x^2 + xy\Big|_{x=0}^{1} \\
&= \frac{1}{2} + y. \\
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{2x+2y}{1+2y}, \text{ if } x, y \in (0,1).
\end{aligned}$$

b. $P[B] = 1 - P[B^c]$, where $B^c = \{X + Y < 1/4\}$.

$$\begin{aligned}
P[B^c] &= \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}-y} x + y\,dxdy \\
&= \int_0^{\frac{1}{4}} \left(\frac{1}{2}x^2 + xy\right)_{x=0}^{\frac{1}{4}-y} dy \\
&= \int_0^{\frac{1}{4}} \frac{1}{32} - \frac{1}{2}y^2\,dy \\
&= \frac{1}{32}y - \frac{1}{6}y^3\Big|_{y=0}^{\frac{1}{4}} \\
&= \frac{1}{192}. \\
P[B] &= \frac{191}{192}.
\end{aligned}$$

Problem 4. Suppose we have two independent random variables X and Y with respective densities

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} y/2, & 0 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $W = X/Y$. Find the density function for W.

Solution: Let $w = g(x, y) = \frac{x}{y}$ and $z = h(x, y) = y$. The transformation of variables can be also written as $x = wz$ and $y = z$.

$$f_{W,Z}(w, z) = f_{X,Y}(x, y) \cdot |J|,$$

where

$$J = \left| \begin{bmatrix} \frac{dx}{dz} & \frac{dx}{dw} \\ \frac{dy}{dz} & \frac{dy}{dw} \end{bmatrix} \right| = \left| \begin{bmatrix} w & z \\ 1 & 0 \end{bmatrix} \right| = -z.$$

Hence,

$$\begin{aligned}
f_{W,Z}(w, z) &= f_{X,Y}(wz, z)|z| \\
&= \frac{z^2}{2} e^{-wz} \text{ if } w > 0 \text{ and } 0 < z < 2. \\
f_W(w) &= \int_0^2 \frac{z^2}{2} e^{-wz}\, dz \\
&= \frac{1}{2} \left[-\frac{z^2}{w} e^{-wz} \Big|_{z=0}^2 + \int_0^2 \frac{2z}{w} e^{-wz} dz \right], \text{ integration by parts} \\
&= -\frac{2}{w} e^{-2w} + \int_0^2 \frac{z}{w} e^{-wz} dz \\
&= -\frac{2}{w} e^{-2w} + \left[-\frac{z}{w^2} e^{-wz} \Big|_{z=0}^2 + \int_0^2 \frac{1}{w^2} e^{-wz} dz \right], \text{ integration by parts} \\
&= -\frac{2}{w} e^{-2w} - \frac{2}{w^2} e^{-2w} - \frac{1}{w^3} e^{-2w} + \frac{1}{w^3} \\
&= \frac{1}{w^3} \left(1 - e^{-2w} - 2we^{-2w} - 2w^2 e^{-2w} \right), \text{ if } w > 0.
\end{aligned}$$

Problem 5. Let X be a normally distributed random variable with mean 1 and variance 1. Suppose $Y = f(X)$, i.e., Y is a function of X. It is known that $E(Y) = 5$ and $Var(Y) = 29$. Furthermore, $r_{XY} = 1$, i.e., the correlation coefficient between X and Y is 1. Find the function f.

Solution:

Since $r_{XY} = 1$, $Y = f(X) = aX + b$ with probability 1.

$$\begin{aligned}\mathbf{E}[Y] &= \mathbf{E}[aX + b] = a + b = 5 \\ \text{var}(Y) &= a^2\text{var}(X) = a^2 = 29.\end{aligned}$$

We have $a = \sqrt{29}$ and $b = 5 - \sqrt{29}$ and thus $Y = f(X) = \sqrt{29}X + (5 - \sqrt{29})$.

Problem 6. Suppose the two-dimensional ranndom variable (X, Y) has desnity

$$f(x, y) = \begin{cases} x^3 + \dfrac{xy^2}{\alpha}, & 0 < x < 1,\ 0 < y < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Show that the value of α that makes this a valid density function is $\alpha = 8/3$.

b. Compute $P(X + Y \geq 2)$.

Solution:

a.

$$\begin{aligned} &\int_0^2 \int_0^1 x^3 + \frac{1}{\alpha}xy^2\,dxdy = 1 \\ \Rightarrow &\int_0^2 \frac{1}{4}x^4 + \frac{1}{2\alpha}x^2y^2\Big|_{x=0}^1 dy = 1 \\ \Rightarrow &\int_0^2 \frac{1}{4} + \frac{1}{2\alpha}y^2\,dy = 1 \\ \Rightarrow &\frac{1}{4}y + \frac{1}{6\alpha}y^3\Big|_{y=0}^2 = 1 \\ \Rightarrow &\frac{1}{2} + \frac{4}{3\alpha} = 1 \\ \Rightarrow &\alpha = \frac{8}{3}. \end{aligned}$$

b.

$$
\begin{aligned}
P[X+Y \geq 2] &= \int_0^1 \int_{2-x}^2 x^3 + \frac{3}{8}xy^2 \, dydx \\
&= \int_0^1 x^3y + \frac{1}{8}xy^3 \Big|_{y=2-x}^2 \, dx \\
&= \int_0^1 x^4 + \frac{1}{8}x \cdot (2^3 - (2-x)^3) \, dx \\
&= \int_0^1 x^4 + \frac{1}{8}(x^4 - 6x^3 + 12x^2) \, dx \\
&= \frac{1}{5}x^5 + \frac{1}{8}\left(\frac{1}{5}x^5 - \frac{3}{2}x^4 + 4x^3\right)\Big|_{x=0}^1 \\
&= \frac{43}{80}.
\end{aligned}
$$

Problem 7. Let Ω be an uncountable set and let

$$F = \left\{E \subseteq \Omega : E \text{ or } \overline{E} \text{ is at most countable}\right\}.$$

Show F is a $\sigma-$algebra (or $\sigma-$field).

Solution: We say a set E is at most countable if E is either finite or countably infinite. In order to verify F is a $\sigma-$algebra, we check the three requirements:

1. Empty set:
   $\phi \in F$, since ϕ is at most countable.

2. Closed under complement:
   If $E \in F$, then either E or $\overline{E}$ is at most countable $\Rightarrow \overline{E} \in F$.

3. Closed under countable unions:
   If $E_i \in F$, $\forall i \in I$, we have to consider 2 cases:
   (a) If E_i is at most countable $\forall i \in I$, then $\cup_i E_i$ is at most countable as well $\Rightarrow \cup_i E_i \in F$.
   (b) If there is at least a $j \in I$ such that E_j is uncountable but $\overline{E}_j$ is at most countable ($E_j \in F$), then $\cap_i \overline{E}_i \subseteq \overline{E}_j$ and thus $\cap_i \overline{E}_i$ is at most countable $\Rightarrow \cup_i E_i \in F$ since $\cap_i \overline{E}_i$ is the complement of $\cup_i E_i$.