EE 503

Homework 7 Solution

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Problem 1. Find the density of $Y = \sin^{-1}(X)$ when

a. X is uniform on [0,1].

b. X is uniform on [-1,1].

Solution: $Y = g(X) = \sin^{-1}(X)$, where $g : [-1, 1] \to [-\pi/2, \pi/2]$.

a. Note that X has cdf $F_X(x) = x$, if $0 \le x \le 1$. Therefore,

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P[\sin^{-1}(X) \le y] \\ &= \begin{cases} P[X \le \sin(y)], & \text{if } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \\ 1, & \text{if } y > \frac{\pi}{2}, \\ 0, & \text{if } y < -\frac{\pi}{2}, \end{cases} \\ &= \begin{cases} F_X(\sin(y)), & \text{if } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \\ 1, & \text{if } y > \frac{\pi}{2}, \\ 0, & \text{if } y < -\frac{\pi}{2}, \end{cases} \\ &= \begin{cases} \sin(y), & \text{if } 0 \le y \le \frac{\pi}{2}, \\ 1, & \text{if } y > \frac{\pi}{2}, \\ 0, & \text{if } y < 0. \end{cases} \end{aligned}$$

And,

$$f_Y(y) = \begin{cases} \cos(y), & \text{if } 0 \le y \le \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

b. Similarly, we know that X has cdf $F_X(x) = \frac{x}{2} + \frac{1}{2}$, if $-1 \le x \le 1$.

Therefore,

$$F_Y(y) = \begin{cases} F_X(\sin(y)), & \text{if } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \\ 1, & \text{if } y > \frac{\pi}{2}, \\ 0, & \text{if } y < -\frac{\pi}{2}, \end{cases}$$
$$= \begin{cases} \frac{\sin(y)}{2} + \frac{1}{2}, & \text{if } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \\ 1, & \text{if } y > \frac{\pi}{2}, \\ 0, & \text{if } y < -\frac{\pi}{2}. \end{cases}$$

And,

$$f_Y(y) = \begin{cases} \frac{\cos(y)}{2}, & \text{if } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2. Let $p_{ik} = P(X = i\ Y = k)$. You are given the following joint probability distribution of the discrete random variable $(X,\ Y)$: $p_{11} = 1/12$, $p_{12} = 0$, $p_{13} = 1/18$, $p_{21} = 1/6$, $p_{22} = 1/9$, $p_{23} = 1/4$, $p_{31} = 0$, $p_{32} = 1/5$, $p_{33} = 2/15$. Find all marginal distributions.

Solution:
We have the joint pmf $p_{X,Y}(x, y) = p_{xy}$ and we would like to compute the marginal pmfs $p_X(x)$ and $p_Y(y)$, where

$$p_X(x) = \sum_y p_{X,Y}(x, y),$$
$$p_Y(y) = \sum_x p_{X,Y}(x, y).$$

Hence,

$$p_X(1) = \frac{5}{36}, p_X(2) = \frac{19}{36}, p_X(3) = \frac{1}{3},$$
$$p_Y(1) = \frac{1}{4}, p_Y(2) = \frac{14}{45}, p_Y(3) = \frac{79}{180}.$$

Problem 3. Consider the joint density for random variables X and Y

$$f(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine if X and Y are independent.

Solution:
Given the joint pdf $f_{X,Y}(x,y) = x+y$ if $x,y \in (0,1)$, we have the marginal pdfs $f_X(x)$ and $f_Y(y)$ as

$$f_X(x) = \int_0^1 x+y\,dy = \left(xy+\frac{1}{2}y^2\right)\Big|_{y=0}^{1} = x+\frac{1}{2},$$
$$f_Y(y) = \int_0^1 x+y\,dx = \left(\frac{1}{2}x^2+xy\right)\Big|_{x=0}^{1} = y+\frac{1}{2}.$$

Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, the two random variables X and Y are not independent.

Problem 4. Consider the joint density for random variables X and Y

$$f(x,y) = \begin{cases} C(x+2y), & 0<x<2,\ 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the value of C.

b. Find the marginal density of X, $f_X(x)$.

c. Find the joint cdf of X and Y.

d. Find the pdf of the random variable $Z = 9/(X+1)^2$.

Solution:

a.

$$\begin{aligned}
&\int_0^2 \int_0^1 C(x+2y)\,dydx = 1\\
\Rightarrow &\int_0^2 C(x+1)dx = 1\\
\Rightarrow &\, 4C = 1\\
\Rightarrow &\, C = \frac{1}{4}.
\end{aligned}$$

b.

$$\begin{aligned}
f_X(x) &= \int_0^1 \frac{1}{4}(x+2y)\,dy,\ \text{if } 0 < x < 2,\\
&= \frac{1}{4}(x+1),\ \text{if } 0 < x < 2.
\end{aligned}$$

c.

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,dtds,\\
&= \int_0^x \int_0^y \frac{1}{4}(s+2t)\,dtds,\ \text{if } 0 < x < 2, 0 < y < 1,\\
&= \frac{1}{8}x^2y + \frac{1}{4}xy^2,\ \text{if } 0 < x < 2, 0 < y < 1.
\end{aligned}$$

To be complete, we write

$$F_{X,Y}(x,y) = \begin{cases}
\frac{1}{8}x^2y + \frac{1}{4}xy^2, & \text{if } 0 < x < 2, 0 < y < 1,\\
\frac{1}{2}y + \frac{1}{2}y^2, & \text{if } x \geq 2, 0 < y < 1,\\
\frac{1}{8}x^2 + \frac{1}{4}x, & \text{if } y \geq 1, 0 < x < 2,\\
1, & \text{if } x \geq 2, y \geq 1\\
0, & \text{otherwise.}
\end{cases}$$

d. We have $F_X(x) = \frac{1}{8}x^2 + \frac{1}{4}x$, if $x \in (0,2)$. Hence the cdf of Z is

$$\begin{aligned}
F_Z(z) &= P[Z \le z] \\
&= P\left[\frac{9}{(X+1)^2} \le z\right] \\
&= P\left[(X+1)^2 \ge \frac{9}{z}\right] \\
&= P\left[X+1 \ge \frac{3}{\sqrt{z}}\right], \text{ if } z > 0, \text{we can take square root since } X+1 \in (1,3) \\
&= P\left[X \ge \frac{3}{\sqrt{z}} - 1\right], \text{ if } z > 0 \\
&= 1 - F_X\left(\frac{3}{\sqrt{z}} - 1\right), \text{ if } z > 0 \\
&= \begin{cases} \frac{9}{8} - \frac{9}{8z}, & \text{if } z \in [1,9], \\ 1, & \text{if } z > 9, \\ 0, & \text{if } z < 1. \end{cases}
\end{aligned}$$

And the pdf of Z,

$$f_Z(z) = \begin{cases} \frac{9}{8z^2}, & \text{if } z \in [1,9], \\ 0, & \text{otherwise.} \end{cases}$$