EE 503

Homework 6 Solution

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Problem 1. Let X be a geometric random variable, i.e.,

$$P(X = k) = (1-p)^{k-1}p,\ \ k = 1, 2, \ldots$$

Find the conditional distribution function $F_X(x|A)$ where A is the event

a. $A = \{X > s\}$.

b. $A = \{X < s\}$.

c. $A = \{X \text{ is even}\}$.

Solution: Notice that for $X \sim \text{Geometric}(p)$, it has cdf $F_X(x)$

$$\begin{aligned} F_X(x) &= \sum_{k=1}^{x} p(1-p)^{k-1} \\ &= \frac{p[1-(1-p)^x]}{1-(1-p)} \\ &= 1-(1-p)^x, \text{if } x \in \mathbb{N}. \end{aligned}$$

In general, its cdf $F_X(x)$ is well-defined on the entire real line,

$$F_X(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor}, & \text{if } x \in [1, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

However, we usually deal with $x \in [1, \infty)$ only since it is trivial to consider a geometric random variable less than 1. Then, we can compute the conditional cdf of X for a given event A.

a. $A = \{X > s\}$
   If $s < 1$, $P[X > s] = 1$, i.e. the event $\{X > s\}$ always occurs. Hence, $F_X(x|\{X > s\}) = F_X(x)$ if $s < 1$. Next, we consider the case where

$s \geq 1$.

$$\begin{aligned} F_X(x|\{X > s\}) &= \frac{P[X \leq x, X > s]}{P[X > s]} \\ &= \frac{P[s < X \leq x]}{P[X > s]}, \text{ if } x > s \\ &= \frac{F_X(x) - F_X(s)}{1 - F_X(s)}, \text{ if } x > s \\ &= \frac{1 - (1-p)^{\lfloor x \rfloor} - 1 + (1-p)^{\lfloor s \rfloor}}{1 - 1 + (1-p)^{\lfloor s \rfloor}} \text{ if } x > s \geq 1 \\ &= \frac{(1-p)^{\lfloor s \rfloor} - (1-p)^{\lfloor x \rfloor}}{(1-p)^{\lfloor s \rfloor}} \text{ if } x > s \geq 1. \end{aligned}$$

Note that $P[X \leq x, X > s] = 0$, if $x \leq s$.

b. $A = \{X < s\}$

$$F_X(x|\{X < s\}) = \begin{cases} \frac{P[X \leq x, X < s]}{P[X < s]}, & \text{if } s > 1 \\ 0, & \text{otherwise, since } P[X < s] = 0 \text{ if } s \leq 1. \end{cases}$$

We assume the non-trivial case where $s > 1$ from now on.

$$\begin{aligned} F_X(x|\{X < s\}) &= \frac{P[X \leq x, X < s]}{P[X < s]} \\ &= \begin{cases} \frac{P[X \leq x]}{P[X < s]}, & \text{if } x < s \\ \frac{P[X < s]}{P[X < s]}, & \text{if } x \geq s \end{cases} \\ &= \begin{cases} \frac{1-(1-p)^{\lfloor x \rfloor}}{1-(1-p)^{\lceil s \rceil - 1}}, & \text{if } x < s, \\ 1, & \text{if } x \geq s. \end{cases} \end{aligned}$$

Note that $P[X < s] = 1 - (1-p)^{\lceil s \rceil - 1}$.

c. $A = \{X \text{ is even}\}$

$$\begin{aligned} F_X(x|\{X \text{ is even}\}) &= \frac{P[X \leq x, X \text{ is even}]}{P[X \text{ is even}]} \\ &= \frac{\sum_{k=1}^{\lfloor x/2 \rfloor} p(1-p)^{2k-1}}{\sum_{k=1}^{\infty} p(1-p)^{2k-1}}, \end{aligned}$$

where the geometric series

$$\sum_{k=1}^{\infty} p(1-p)^{2k-1} = \frac{p(1-p)}{1-(1-p)^2},$$

and the sum of first n terms of the geometric series

$$\sum_{k=1}^{n} p(1-p)^{2k-1} = \frac{p(1-p)[1-(1-p)^{2n}]}{1-(1-p)^2}.$$

Hence,

$$F_X(x|\{X \text{ is even}\}) = 1-(1-p)^{2\lfloor x/2 \rfloor}.$$

In part a. and part b. of problem 1, it is totally fine if you just consider $x, s \in \mathbb{N}$, so that the floor and ceiling function do not matter, i.e. $x = \lfloor x \rfloor = \lceil x \rceil$.

Problem 2. Suppose $P(X=k) = \beta(1-\beta)^{k-1},\ 0<\beta<1,\ k=1,\ 2,\ 3, \ldots.$ Find Var(X).

Solution:
We have $X \sim \text{Geometric}(\beta)$ and we know that $\mathbf{E}[X] = \frac{1}{\beta}$ and $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$. In this problem, instead of computing $\mathbf{E}[X^2]$ directly, it is better to compute $\mathbf{E}[(X+1)X]$ with a trick.

$$\begin{aligned}
\mathbf{E}[(X+1)X] &= \sum_{x=1}^{\infty}(x+1)x\beta(1-\beta)^{x-1} \\
&= \beta\sum_{x=1}^{\infty}(x+1)x\alpha^{x-1}, \text{ where } \alpha = 1-\beta, 0<\alpha<1.
\end{aligned}$$

Observe that $(x+1)x\alpha^{x-1} = \frac{d^2}{d\alpha^2}\alpha^{x+1}$; thus

$$\begin{aligned}
\mathbf{E}[(X+1)X] &= \beta \sum_{x=1}^{\infty} \frac{d^2}{d\alpha^2}\alpha^{x+1} \\
&= \beta \frac{d^2}{d\alpha^2}\left(\sum_{x=1}^{\infty}\alpha^{x+1}\right), \text{since } (*) \\
&= \beta \frac{d^2}{d\alpha^2}\left(\frac{\alpha^2}{1-\alpha}\right) \\
&= \beta \frac{d}{d\alpha}\left(\frac{2\alpha(1-\alpha)+\alpha^2}{(1-\alpha)^2}\right) \\
&= \beta \frac{d}{d\alpha}\left(\frac{2\alpha-\alpha^2}{(1-\alpha)^2}\right) \\
&= \beta \cdot \frac{(2-2\alpha)(1-\alpha)^2 + 2(1-\alpha)(2\alpha-\alpha^2)}{(1-\alpha)^4} \\
&= \beta \cdot \frac{2(1-\alpha)^3 + 2(1-\alpha)\alpha(2-\alpha)}{(1-\alpha)^4} \\
&= \beta \cdot \frac{2\beta^3 + 2\beta(1-\beta)(1+\beta)}{\beta^4}, \text{ replace } \alpha \text{ with } 1-\beta \\
&= \frac{2}{\beta^2}.
\end{aligned}$$

$(*)$ In calculus, we learned that we may interchange the order of infinite sum and differentiation operator since $f_n(\alpha) = \sum_{x=1}^{n}\alpha^{x+1}$ is twice differentiable and $f_n(\alpha)$ converges uniformly to $f(\alpha) = \frac{\alpha^2}{1-\alpha}$ as $n \to \infty$ for all $\alpha \in (0,1)$. Finally,

$$\begin{aligned}
\operatorname{var}(X) &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\
&= \left(\frac{2}{\beta^2} - \frac{1}{\beta}\right) - \frac{1}{\beta^2} \\
&= \frac{1-\beta}{\beta^2}.
\end{aligned}$$

Problem 3. The continuous random variable X has pdf

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find Var(X).

Solution: We have $X \sim$ Exponential(2) with $\mathbf{E}[X] = \int_0^\infty x 2e^{-2x} dx = \frac{1}{2}$.

$$\begin{aligned}\mathbf{E}[X^2] &= \int_0^\infty x^2 2e^{-2x}\, dx \\ &= \left(-e^{-2x}x^2\right)_{x=0}^{\infty} - \int_0^\infty -2xe^{-2x}\, dx \\ &= 0 + \int_0^\infty 2xe^{-2x} dx \\ &= \frac{1}{2}.\end{aligned}$$

Hence, $\mathrm{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{1}{4}$.

Problem 4. The continuous random variable X has pdf

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = X^4$. Find Var(Y).

Remark: You do not need to find the pdf of Y to work this problem.

Solution:

$$
\begin{aligned}
\mathbf{E}[Y] &= \int_0^1 x^4 \cdot 3x^2 dx \\
&= \frac{3}{7}x^7\Big|_{x=0}^1 \\
&= \frac{3}{7}. \\
\mathbf{E}[Y^2] &= \int_0^1 x^8 \cdot 3x^2 dx \\
&= \frac{3}{11}x^{11}\Big|_{x=0}^1 \\
&= \frac{3}{11}. \\
\mathrm{var}(Y) &= \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 \\
&= \frac{3}{11} - \frac{9}{49} \approx 0.0891.
\end{aligned}
$$

Problem 5. A man is saving up to buy a new car at a cost of N units of money. He starts with k units of money $(0 < k < N)$ and tries to win the remainder with the following gamble with his bank manager. He tosses a fair coin; if it turns up heads the bank manager pays him one unit of money, but if it comes up tails then he pays the manager one unit of money. He keeps tossing the coin and playing this game until either he has won enough units of money to buy the car or he loses his k units of money (goes bankrupt).

Let A_k denote the event that he is eventually bankrupt after his initial capital was k units. Let $p_k = P(A_k)$.

a. Show $p_k = \frac{1}{2}(p_{k+1} + p_{k-1})$ if $0 < k < N$.

b. The result in part (a) is a linear difference equation subject to the boundary conditions $p_0 = 1,\ p_N = 0$. Solve this difference equation for p_k. Hint: If you have not solved this type of equation analytically before then you can instead proceed directly as follows: First let $b_k = p_k - p_{k-1}$. Show $b_k = b_{k-1}$ and thus $b_k = b_1$ for all k. Continue from here.

Solution:

a. Using total probability theory, we have

$$\begin{aligned} P(A_k) &= P(A_k|\{\text{first toss head}\})P(\{\text{first toss head}\}) \\ &\quad + P(A_k|\{\text{first toss tail}\})P(\{\text{first toss tail}\}) \\ &= p_{k+1} \cdot \frac{1}{2} + p_{k-1} \cdot \frac{1}{2}. \end{aligned}$$

Hence, $p_k = \frac{1}{2}(p_{k+1} + p_{k-1})$.

b. Let $b_k = p_k - p_{k-1}$, $\forall k \in \{1, 2, \cdots, N\}$. Notice that $b_k = b_{k-1}$ from part a. and

$$\sum_{k=1}^{N} b_k = p_N - p_0 = 0 - 1 = -1.$$

We have $b_k = -\frac{1}{N}$, $\forall k \in \{1, 2, \cdots, N\}$. Finally,

$$\begin{aligned} b_1 = p_1 - p_0 &= -\frac{1}{N} \Rightarrow p_1 = \frac{N-1}{N}, \\ b_2 = p_2 - p_1 &= -\frac{1}{N} \Rightarrow p_2 = \frac{N-2}{N}, \\ &\vdots \\ b_k = p_k - p_{k-1} &= -\frac{1}{N} \Rightarrow p_k = \frac{N-k}{N}, \forall k \in \{1, 2, \cdots N\}. \end{aligned}$$

Problem 6. Suppose the discrete random variable X has probability mass function

$$P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \ldots, \ \lambda \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the moment generating function of X and use it to compute $E[X]$ and $Var(X)$.

Solution: We have $X \sim \text{Poisson}(\lambda)$ and the moment generating function of

X is $M_X(s) = \mathbf{E}[e^{sX}]$.

$$\begin{aligned}
\mathbf{E}[e^{sX}] &= \sum_{x=0}^{\infty} e^{sx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^s)^x}{x!} \\
&= \sum_{x=0}^{\infty} \frac{e^{-\lambda e^s}(\lambda e^s)^x}{x!} \cdot e^{-\lambda(1-e^s)} \\
&= e^{\lambda(e^s-1)}, \text{ since the first term is summing up the pmf of Poisson}(\lambda e^s),
\end{aligned}$$

We can compute every moment from mgf via differentiation,

$$\begin{aligned}
\mathbf{E}[X] &= \frac{d}{ds} M_X(s)\Big|_{s=0} \\
&= e^{\lambda(e^s-1)} \cdot \lambda e^s\Big|_{s=0} \\
&= \lambda. \\
\mathbf{E}[X^2] &= \frac{d^2}{ds^2} M_X(s)\Big|_{s=0} \\
&= \left[e^{\lambda(e^s-1)} \cdot (\lambda e^s)^2 + e^{\lambda(e^s-1)} \cdot \lambda e^s\right]_{s=0} \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Hence, $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \lambda$.