EE 503

Homework 5 Solution

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Problem 1. Suppose $X \sim N(0,\ 1)$. Let $Y = X^2$. Find the pdf of Y.

Solution: To find the pdf of Y, we first compute the cdf of Y and notice that the event $\{Y \leq y\}$ occurs when $\{X^2 \leq y\}$ or equivalently when $\{-\sqrt{y} \leq X \leq \sqrt{y}\}$ if $y \geq 0$.

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[-\sqrt{y} \leq X \leq \sqrt{y}],\ \text{if } y \geq 0 \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}),\ \text{if } y \geq 0. \end{aligned}$$

Hence,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y}),\ \text{if } y \geq 0 \\ &= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) - \frac{-1}{2\sqrt{y}} f_X(-\sqrt{y}),\ \text{if } y \geq 0 \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}},\ \text{if } y \geq 0. \end{aligned}$$

Notice that $f_Y(y) = 0$ if $y < 0$.

Problem 2. Suppose X is uniform over the interval (-1, 1). Let $W = |X|$. Find the pdf of W.

Solution:

$$\begin{aligned} F_W(w) &= P[W \leq w] \\ &= P[|X| \leq w] \\ &= \begin{cases} P[-w \leq X \leq w], & \text{if } 1 \geq w \geq 0, \\ 1, & \text{if } w > 1, \\ 0, & \text{if } w < 0. \end{cases} \end{aligned}$$

We know that $P[-w \le X \le w] = w$ if $1 \ge w \ge 0$. Hence,

$$
\begin{aligned}
f_W(w) &= \frac{d}{dw} F_W(w) \\
&= \begin{cases} 1, & \text{if } 1 \ge w \ge 0, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

We see that $W \sim \text{Uniform}((0,1))$.

Problem 3. Suppose $P(X = k) = \beta(1-\beta)^{k-1},\ 0 < \beta < 1,\ k = 1,\ 2,\ 3, ...$ Find E(X).

Solution: Since

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{x=1}^{\infty} x\beta(1-\beta)^{x-1} = \beta + 2\beta(1-\beta) + 3\beta(1-\beta)^2 + \cdots \\
(1-\beta)\mathbf{E}[X] &= \beta(1-\beta) + 2\beta(1-\beta)^2 + \cdots ,
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathbf{E}[X] - (1-\beta)\mathbf{E}[X] &= \beta + \beta(1-\beta) + \beta(1-\beta)^2 + \cdots \\
&= \frac{\beta}{1-(1-\beta)} = 1.
\end{aligned}
$$

Hence, $[1-(1-\beta)]\mathbf{E}[X] = 1$ and thus $\mathbf{E}[X] = \frac{1}{\beta}$.

Problem 4. The continuous random variable X has pdf

$$
f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}
$$

Find E(X).

Solution: We are going to use integration by parts $\int u dv = (uv) - \int v du$ where $u = 2x, du = 2dx, v = -\frac{1}{2}e^{-2x}, dv = e^{-2x}dx$ to compute the mean of

$X \sim$ Exponential(2).

$$\begin{aligned}
\mathbf{E}[X] &= \int_0^\infty 2xe^{-2x}dx \\
&= \left(-xe^{-2x}\right)_{x=0}^\infty - \int_0^\infty -e^{-2x}dx, \text{ integration by parts} \\
&= \int_0^\infty e^{-2x}dx \\
&= \frac{-1}{2}e^{-2x}\Big|_{x=0}^\infty \\
&= \frac{1}{2}.
\end{aligned}$$

Problem 5. The continuous random variable X has pdf

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = X^4$. Find E(Y).

Remark: You do not need to find the pdf of Y to work this problem.
Solution:

$$\begin{aligned}
\mathbf{E}[Y] &= \mathbf{E}[X^4] \\
&= \int_{-\infty}^\infty x^4 f(x)dx \\
&= \int_0^1 3x^6 dx \\
&= \frac{3}{7}x^7\Big|_{x=0}^1 \\
&= \frac{3}{7}.
\end{aligned}$$

Problem 6. Text 4.34. The Pareto random variable X has cdf

$$F_X(x) = \begin{cases} 0, & x < x_m \\ 1 - \dfrac{x_m^\alpha}{x^\alpha}, & x \ge x_m, \quad x_m > 0, \ \alpha > 0. \end{cases}$$

a. Find and plot the pdf of X.

b. Find and plot $F_X(x|X > t)$.

c. Find and plot $f_X(x|X > t)$.

Solution:

a. The pdf of X is

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \begin{cases} 0, & \text{if } x < x_m, \\ \frac{\alpha x_m^\alpha}{x^{\alpha+1}}, & \text{if } x \geq x_m. \end{cases} \end{aligned}$$

b.

$$\begin{aligned} F_X(x|X > t) &= \frac{P[X \leq x, X > t]}{P[X > t]} \\ &= \frac{P[t < X \leq x]}{P[X > t]}, \text{ if } x > t \\ &= \begin{cases} \frac{F_X(x) - F_X(t)}{1 - F_X(t)}, & \text{if } x > t, \\ 0, & \text{if } x \leq t. \end{cases} \end{aligned}$$

If $t < x_m$, then $F_X(t) = 0$ and $F_X(x|X > t) = 1 - \frac{x_m^\alpha}{x^\alpha}$, if $x \geq x_m$.
If $t \geq x_m$, then

$$\begin{aligned} F_X(x|X > t) &= \frac{1 - \frac{x_m^\alpha}{x^\alpha} - 1 + \frac{x_m^\alpha}{t^\alpha}}{1 - \left(1 - \frac{x_m^\alpha}{t^\alpha}\right)} \\ &= 1 - \frac{t^\alpha}{x^\alpha}, \text{ if } x > t. \end{aligned}$$

To conclude,

$$F_X(x|X > t) = \begin{cases} 1 - \frac{x_m^\alpha}{x^\alpha}, & \text{if } x \geq x_m > t, \\ 1 - \frac{t^\alpha}{x^\alpha}, & \text{if } x > t \geq x_m, \\ 0, & \text{otherwise.} \end{cases}$$

c. Based on the result in part b., we have

$$
\begin{aligned}
f_X(x|X > t) &= \frac{d}{dx} F_X(x|X > t) \\
&= \begin{cases} \frac{\alpha x_m^\alpha}{x^{\alpha+1}}, & \text{if } x \geq x_m > t, \\ \frac{\alpha t^\alpha}{x^{\alpha+1}}, & \text{if } x > t \geq x_m, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$