EE 503

Homework 4 Solution

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Problem 1. Suppose that f and g are probability density functions (pdf's) defined on the same interval $[a,\ b]$.

a. Show that $f + g$ is not a valid pdf on the same interval.

b. For any number $\beta,\ 0 < \beta < 1,$ show that $\beta f(x) + (1-\beta)g(x)$ is a valid pdf on the interval $[a,\ b]$.

Solution:

a. Since

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx = 2,$$

$f + g$ is not a valid pdf.

b. We notice that if $0 < \beta < 1$, then $\beta f(x) + (1-\beta)g(x) \geq 0$, $\forall x \in [a, b]$ since both f and g are valid pdfs defined on $[a, b]$. In addition,

$$\int \beta f(x) + (1-\beta)g(x)\ dx = \beta + (1-\beta) = 1.$$

Hence, if $0 < \beta < 1$, then $\beta f(x) + (1-\beta)g(x)$ is a valid pdf.

Problem 2. Let X be the life length of an electronic device (measured in hours). Suppose that X is a continuous random variable with *pdf*

$$f(x) = k/x^n,\ 2,000 \leq x \leq 10,000.$$

a. For $n = 2$, determine k.

b. For $n = 3$, determine k.

c. For general n, determine k.

d. For general n, find the probability that the device will reach the end of its life before 5000 hours have elapsed.

Solution:

a.

$$\begin{aligned}&\int_{2000}^{10000} \frac{k}{x^2}\,dx = 1\\ \Rightarrow\ & -\frac{k}{x}\Big|_{x=2000}^{10000} = 1\\ \Rightarrow\ & k = 2500.\end{aligned}$$

b.

$$\begin{aligned}&\int_{2000}^{10000} \frac{k}{x^3}\,dx = 1\\ \Rightarrow\ & -\frac{k}{2x^2}\Big|_{x=2000}^{10000} = 1\\ \Rightarrow\ & k = 8.334 \times 10^6.\end{aligned}$$

c. If $n \neq 1$, $n \in \mathbb{N}$,

$$\begin{aligned}&\int_{2000}^{10000} \frac{k}{x^n}\,dx = 1\\ \Rightarrow\ & -\frac{k}{(n-1)x^{n-1}}\Big|_{x=2000}^{10000} = 1\\ \Rightarrow\ & k = (n-1)\Big/\left(\frac{1}{2000^{n-1}} - \frac{1}{10000^{n-1}}\right).\end{aligned}$$

If $n = 1$, then

$$\begin{aligned}&\int_{2000}^{10000} \frac{k}{x}\,dx = 1\\ \Rightarrow\ & k\ln x\Big|_{x=2000}^{10000} = 1\\ \Rightarrow\ & k = \frac{1}{\ln 5}.\end{aligned}$$

d. If $n \neq 1$, $n \in \mathbb{N}$,

$$\begin{aligned} P[X \leq 5000] &= \int_{2000}^{5000} \frac{k}{x^n}\, dx \\ &= -\frac{k}{(n-1)x^{n-1}}\Big|_{x=2000}^{5000} \\ &= \frac{\frac{1}{2000^{n-1}} - \frac{1}{5000^{n-1}}}{\frac{1}{2000^{n-1}} - \frac{1}{10000^{n-1}}} \\ &= \frac{5^{n-1} - 2^{n-1}}{5^{n-1} - 1}. \end{aligned}$$

If $n = 1$, then

$$\begin{aligned} P[X \leq 5000] &= \frac{1}{\ln 5} \int_{2000}^{5000} \frac{1}{x}\, dx \\ &= \frac{1}{\ln 5} \cdot \ln 2.5 \\ &= 0.5693. \end{aligned}$$

Problem 3. If the random variable K is uniformly distributed over (0, 5), what is the probability that the roots of the equation $4x^2 + 4Kx + K + 2 = 0$ are real.

Solution:
Notice that a quadratic equation $f(x) = ax^2 + bx + c = 0$ has real roots if the **discriminant** $b^2 - 4ac \geq 0$. Therefore,

$$\begin{aligned} &16K^2 - 4 \cdot 4 \cdot (K+2) \geq 0 \\ \Rightarrow\ &K^2 - K - 2 \geq 0 \\ \Rightarrow\ &(K-2)(K+1) \geq 0 \\ \Rightarrow\ &K \geq 2 \text{ or } K \leq -1. \end{aligned}$$

Hence, we have the probability of the equation $4x^2 + 4Kx + K + 2 = 0$ having real roots as $P[K \geq 2] = 3/5$ since $K \sim \text{Uniform}((0,5))$.

Problem 4. Suppose that the random variable X has possible values 1, 2, 3,... and that

$$P(X = r) = k(1-\beta)^{r-1},\ 0 < \beta < 1.$$

a. Determine the constant k.

b. Find the *mode* of this distribution, i.e., find that value of r which makes $P(X = r)$ maximum.

Solution:

a.

$$\sum_{r=1}^{\infty} k(1-\beta)^{r-1} = 1$$
$$\Rightarrow \frac{k}{1-(1-\beta)} = 1, \text{ formula of infinite geometric series}$$
$$\Rightarrow k = \beta.$$

This is a Geometric random variable with parameter β.

b. The mode of a discrete distribution is a set of values that maximize the probability mass function (pmf). We have the pmf of X as

$$p_X(x) = \beta(1-\beta)^{x-1}, \text{ for } x \in \mathbb{N}.$$

We observe that $p_X(1) = \beta$, $p_X(2) = \beta(1-\beta)$, $p_X(3) = \beta(1-\beta)^2$, etc. Since $0 < (1-\beta) < 1$, the mode of $p_X(x)$ is $x = 1$.

Problem 5. Say someone has two coins and tells you that one of the coins has 2 heads and the other coin is fair (you cannot see the coins).

a. Suppose the person flips the coins one after another. Let $X = 1$ if the first toss is heads and $X = 0$ if the first toss is tails. Let $Y = 1$ if the second toss is heads and let $Y = 0$ if the second toss is tails. Are X and Y independent random variables? Explain your answer.

b. Suppose the person flips just one of the coins and it turns up heads. Based only on this information can you say which coin is most likely the fair coin? If so, justify your answer. If not, explain why.

Solution:

a. The two random variables X and Y are independent if $P[X = x, Y = y] = P[X = x]P[Y = y]$, $\forall x, y \in \{0, 1\}$. Notice that $P[X = 0, Y = 0] = 0$. However, using total probability,

$$\begin{aligned} P[X = 0] &= P[X = 0|\text{1st coin is fair}]P[\text{1st coin is fair}] \\ &\quad + P[X = 0|\text{2nd coin is fair}]P[\text{2nd coin is fair}] \\ &= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

And similarly, $P[Y = 0] = \frac{1}{4}$. Since there is a case where

$$P[X = 0, Y = 0] = 0 \neq P[X = 0]P[Y = 0] = \frac{1}{16},$$

X and Y are not independent random variables.

b. We compute the conditional probability that the flipped coin is fair given the outcome is a head using Bayes' rule,

$$\begin{aligned} P[\text{fair coin}|\text{H}] &= \frac{P[\text{H}|\text{fair coin}]P[\text{fair coin}]}{P[\text{H}]} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{1}{3}. \end{aligned}$$

Hence, $P[\text{unfair coin}|\text{H}] = 2/3$. We conclude that the coin been flipped is more likely to be the unfair coin.

Problem 6. Text 4.17 (modified). A random variable X has pdf

$$f(x) = \begin{cases} c\left(1 - x^2\right), & -1 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find c.

b. Find the cumulative distribution function (cdf) of X.

c. Find $P(0.25 < X < 0.5)$.

Solution:

a.

$$\int_{-1}^{1} c(1-x^2)\,dx = 1$$
$$\Rightarrow cx - \frac{1}{3}cx^3\Big|_{x=-1}^{1} = 1$$
$$\Rightarrow c = \frac{3}{4}.$$

b.

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \begin{cases} \int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt, & \text{if } -1 \le x \le 1 \\ 0, & \text{if } x < -1 \\ 1, & \text{if } x > 1 \end{cases} \\ &= \begin{cases} \frac{3}{4}(x - \frac{1}{3}x^3) + \frac{1}{2}, & \text{if } -1 \le x \le 1 \\ 0, & \text{if } x < -1 \\ 1, & \text{if } x > 1 \end{cases} \\ &= \begin{cases} -\frac{1}{4}x^3 + \frac{3}{4}x + \frac{1}{2}, & \text{if } -1 \le x \le 1 \\ 0, & \text{if } x < -1 \\ 1, & \text{if } x > 1. \end{cases} \end{aligned}$$

c.

$$\begin{aligned} P\left[\frac{1}{4} < X < \frac{1}{2}\right] &= \int_{1/4}^{1/2} \frac{3}{4}(1-t^2)\,dt \\ &= \frac{3}{4}\left(t - \frac{1}{3}t^3\right)\Big|_{t=\frac{1}{4}}^{\frac{1}{2}} \\ &= \frac{41}{256}. \end{aligned}$$