EE 503

Homework 3 Solution

Instructor: Christopher Wayne Walker, TA: James Huang

Problem 1. Suppose $A \subset B$ with $P(A) = 1/4$ and $P(B) = 1/3$.

a. Find $P(A|B)$.

b. Find $P(B|A)$.

Solution:

a. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{3}{4}$.

b. $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$.

Problem 2. Show that $P(AB|C) = P(A|BC)P(B|C)$.

Solution:

$$\begin{aligned} P(AB|C) &= \frac{P(ABC)}{P(C)} \\ &= \frac{P(A|BC)P(BC)}{P(C)} \\ &= P(A|BC)P(B|C). \end{aligned}$$

Problem 3. Show that $P(ABC) = P(A|BC)P(B|C)P(C)$.

Solution:

$$\begin{aligned} P(ABC) &= P(A|BC)P(BC) \\ &= P(A|BC)P(B|C)P(C). \end{aligned}$$

Problem 4. Box 1 contains 1 White and 999 Red balls and Box 2 contains 1 Red and 999 White balls. Suppose someone picks a box at random and then selects a ball from that box and the ball picked is Red. Find the probability the Red ball came from Box 1.

Solution:

$$\begin{aligned} P(\{\text{From Box 1}\}|\{\text{Red}\}) &= \frac{P(\{\text{Red}\}|\{\text{From Box 1}\})P(\{\text{From Box 1}\})}{P(\{\text{Red}\})} \\ &= \frac{0.999 \times 0.5}{0.999 \times 0.5 + 0.001 \times 0.5} \\ &= 0.999. \end{aligned}$$

Problem 5. Suppose there are events A and B such that

$$P(B) = 0.2,\ P\left(B|A\right) = 0.4,\ P\left(A \cup B\right) = 0.5.$$

Find $P\left(A|B\right)$.

Solution: Since $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A\cap B)}{P(A\cap B)+P(A\cap B^c)} = 0.4$, and $P(A\cap B^c) = P(A \cup B) - P(B) = 0.5 - 0.2 = 0.3$. We then get $P(A \cap B) = 0.2$. Hence,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.2} = 1.$$

Problem 6. Suppose A and B are events such that $A \cap B = \phi$. Can A and B be independent? If not, explain why not. If so, explain how.

Solution: Events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$. If $A \cap B = \phi$, then $P(A \cap B) = 0$. Therefore, A and B are independent if and only if $P(A)P(B) = 0$ in this case. Hence, they can be independent if $P(A) = 0$ or $P(B) = 0$.

Problem 7. Suppose you have R red balls and B blue balls in a container. You pick a ball at random and then replace that ball and C balls of the same color back into the container. Now, pick a ball a second time. Find the probability that the second choice is red.

Solution: We use total probability when solving this problem.

$$\begin{aligned} P(\{\text{2nd is Red}\}) &= P(\{\text{2nd is Red}\}|\{\text{1st is Red}\})P(\{\text{1st is Red}\}) \\ &\quad + P(\{\text{2nd is Red}\}|\{\text{1st is Blue}\})P(\{\text{1st is Blue}\}) \\ &= \frac{R+C}{R+B+C} \cdot \frac{R}{R+B} + \frac{R}{R+B+C} \cdot \frac{B}{R+B} \\ &= \frac{R}{R+B}. \end{aligned}$$

Problem 8. Suppose I have two coins. One is fair so $P(H) = P(T) = 1/2$. The other coin has $P(H) = 1/3$ and $P(T) = 2/3$. Suppose I pick one of the coins at random and give it to you. You flip the coin twice and it comes up heads both times. Based on this information find the probability that I gave you the fair coin.

Solution:

$$\begin{aligned} P(\{\text{Fair}|\text{HH}\}) &= \frac{P(\{\text{HH}\}|\{\text{Fair}\})P(\{\text{Fair}\})}{P(\{\text{HH}\}|\{\text{Fair}\})P(\{\text{Fair}\}) + P(\{\text{HH}\}|\{\text{Unfair}\})P(\{\text{Unfair}\})} \\ &= \frac{1/4 \cdot 1/2}{1/4 \cdot 1/2 + 1/9 \cdot 1/2} \\ &= \frac{9}{13}. \end{aligned}$$

Problem 9. Let X be the set of positive integers. Let S consist of all the subsets E of X such that either E or $\overline{E}$ is finite. Show that S is an algebra but not a σ-algebra.

Solution:
The difference between an algebra and σ-algebra is that algebra is closed under finite unions while σ-algebra is closed under countable (possibly infinite) unions. We notice that $\phi \in S$ (also $X \in S$) and $E \in S \Rightarrow \overline{E} \in S$ if either E or $\overline{E}$ is finite. If we let $E_n = \{2n\}$, then $\bigcup_{i=1}^{n} E_i$ is a finite subset of X which contains even numbers up to $2n$. Hence, $\bigcup_{i=1}^{n} E_i \in S$. However, the set of all

even positive integers

$$\bigcup_{i=1}^{\infty} E_i \notin S,$$

since neither $\bigcup_{i=1}^{\infty} E_i$, nor its complement (odd positive integers) is a finite subset of S.