EE 503

Homework 1 Solution

Instructor: Christopher Wayne Walker, TA: James Huang

Problem 1. Two components C_1 and C_2 of a system are tested and declared to be in one of three possible states: F, functioning; R, not functioning but repairable; and W, worthless.

a. Find the sample space for this experiment.

b. What is the set corresponding to the event "none of the components are worthless"?

Solution:

a. The sample space is a set of possible outcomes, $\Omega = \{FF, FR, FW, RF, RR, RW, WF, WR, WW\}$.

b. $A = \{FF, FR, RF, RR\}$.

Problem 2. Three balls numbered 1 to 3 in an urn are drawn at random one at a time until the urn is empty. The sequence of the ball numbers is noted.

a. Find the sample space for this experiment.

b. Find the sets A_k corresponding to the events "ball number k is selected in the kth draw," for $k = 1,\ 2,\ 3$.

c. Find the set $A_1 \cap A_2 \cap A_3$ and describe the event in words.

d. Find the set $A_1 \cup A_2 \cup A_3$ and describe the event in words.

e. Find the set $(A_1 \cup A_2 \cup A_3)^c$ and describe the event in words.

Solution:

a. $\Omega = \{123, 132, 213, 231, 312, 321\}$.

b. $A_1 = \{123, 132\}$, $A_2 = \{123, 321\}$, $A_3 = \{123, 213\}$.

c. The event which ball number k is selected in the kth draw, $\forall k \in \{1, 2, 3\}$ is $A_1 \cap A_2 \cap A_3 = \{1, 2, 3\}$.

d. The event which exists at least one ball number k drawn in the kth draw for some $k \in \{1, 2, 3\}$ is $A_1 \cup A_2 \cup A_3 = \{123, 132, 321, 213\}$.

e. The event which ball number k is not selected in the kth draw, $\forall k \in \{1, 2, 3\}$ is $(A_1 \cup A_2 \cup A_3)^c = \{231, 312\}$.

Problem 3. Let the events A and B have $P(A) = x$, and $P(B) = y$, and $P(A \cap B) = z$. Use Venn diagrams to find the following:

a. $P(A^c \cup B^c)$.

b. $P(A \cap B^c)$.

c. $P(A^c \cup B)$.

d. $P(A^c \cap B^c)$.

Solution:

a. $P(A^c \cup B^c) = 1 - z$.

b. $P(A \cap B^c) = x - z$.

c. $P(A^c \cup B) = 1 - x + z$

d. $P(A^c \cap B^c) = 1 - x - y + z$

Problem 4. Prove the following identities:

a. $\binom{n}{k} = \binom{n}{n-k}$

b. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Solution:

a. By definition,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

b.

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k \cdot (n-1)! + (n-k) \cdot (n-1)!)}{k!(n-k)!} \\ &= \binom{n}{k}.\end{aligned}$$

Problem 5. If $\mathbf{X} = \{1, 2, 3, 4, 5\}$ find the smallest algebra (or field) that contains the sets $\{1\}$ and $\{2,\ 3,\ 5\}$.

Solution:
An algebra (or field) of the set $\mathbf{X}$ needs to satisfy (1) closed under compliments, (2) closed under finite unions (3) it contains the empty set. Hence, the smallest algebra is $\{\emptyset, \{1\}, \{4\}, \{2, 3, 5\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}, \{1, 4\}, \mathbf{X}\}$.

Problem 6. Let $x,\ y$ be points in $\mathbf{R}$. Show that

$$d(x, y) = |x - y|$$

is a metric on $\mathbf{R}$.

Solution:
A metric is a function $d : \mathbf{R} \times \mathbf{R} \to [0, \infty)$ that maps 2 points of a set to non-negative real numbers satisfying $\forall\ x, y, z \in \mathbf{R}$ (1) $d(x, y) \geq 0$, (2) $d(x, y) = 0)$ if and only if $x = y$, (3) $d(x, y) = d(y, x)$, (4) $d(x, z) \leq d(x, y) + d(y, z)$. Since (1) $|x - y| \geq 0$, (2) $|x - y| = 0 \Leftrightarrow x = y$, (3) $|x - y| = |y - x|$ and (4) $|x - z| \leq |x - y| + |y - z|$, $\forall x, y, z \in \mathbf{R}$, we can conclude that $d(\cdot, \cdot) = |\cdot, \cdot|$ is a metric.

Problem 7. Let $a = (x_1,\ y_1),\ b = (x_2,\ y_2)$ be points in $\mathbf{R}^2$. Show that

$$d(a, b) = \max\{|x_1 - x_2|,\ |y_1 - y_2|\}$$

is a metric on $\mathbf{R}^2$. This $d(a, b)$ is often denoted $d_\infty(a, b)$.

Solution:
Similarly, we check whether

$$d(a, b) = \max\{|x_1 - x_2|,\ |y_1 - y_2|\}$$

satisfies the 4 conditions of a metric:

(1) Non-negativity: $\max\{|x_1 - x_2|,\ |y_1 - y_2|\} \geq 0$, $\forall x_1, y_1, x_2, y_2 \in \mathbf{R}$.

(2) Identity: $\max\{|x_1 - x_2|,\ |y_1 - y_2|\} = 0 \Leftrightarrow x_1 = x_2, y_1 = y_2 \Leftrightarrow a = b$.

(3) Symmetry: $\max\{|x_1 - x_2|,\ |y_1 - y_2|\} = \max\{|x_2 - x_1|,\ |y_2 - y_1|\}$.

(4) Triangle inequality:

Suppose we have three points on $\mathbf{R}^2$, $a = (x_1, y_1), b = (x_2, y_2), c = (x_3, y_3)$. Since $|x_1 - x_3| \leq |x_1 - x_2| + |x_2 - x_3|$ and $|y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3|$, we have

$\max\{|x_1 - x_3|,\ |y_1 - y_3|\} \leq \max\{|x_1 - x_2|,\ |y_1 - y_2|\} + \max\{|x_2 - x_3|,\ |y_2 - y_3|\}$.

Problem 8. Let $d(x, y)$ be a metric on $\mathbf{X}$. Show that

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on $\mathbf{X}$.

Solution:
We know that $d(x, y)$ is a metric, so $\forall x, y, z \in \mathbf{X}$,

(1) Non-negativity: $\frac{d(x,y)}{1+d(x,y)} \geq 0$.

(2) Identity: $\frac{d(x,y)}{1+d(x,y)} = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$.

(3) Symmetry: $\frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = 0$ since $d(x, y) = d(y, x)$.

(4) Triangle inequality:

$$\frac{d(x,z)}{1+d(x,z)} \le \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$
$$\Leftrightarrow \frac{d_1}{1+d_1} \le \frac{d_2}{1+d_2} + \frac{d_3}{1+d_3}$$
$$\Leftrightarrow d_1(1+d_2)(1+d_3) \le d_2(1+d_1)(1+d_3) + d_3(1+d_1)(1+d_2)$$
$$\Leftrightarrow d_1 + d_1d_2 + d_1d_3 + d_1d_2d_3 \le d_2 + d_1d_2 + d_2d_3 + d_1d_2d_3 + d_3 + d_1d_3 + d_2d_3 + d_1d_2d_3$$
$$\Leftrightarrow \; d_1 \le d_2 + d_3 + 2d_2d_3 + d_1d_2d_3$$
$$\Leftarrow d_1 \le d_2 + d_3.$$

Problem 9. We know that if we have n students and n chairs we can seat the students in $n!$ ways (just the number of ways of ordering n items). Explain what is wrong with the following argument that derives a different answer.

We first select a student which can be done in n ways. We then select a chair for this student which can also be done in n ways. Thus the first student is selected and seated in n^2 ways. Next, we select the second student in $(n-1)$ ways and her chair in $(n-1)$ ways so the selection and seating of the second student can be done in $(n-1)^2$ ways. Continuing on in this way, we select and seat the n students in $n^2(n-1)^2(n-2)^2 \cdots 3^2 2^2 1^2 = (n!)^2$ ways.

Solution: We count each outcome $n!$ times. Selecting student number 1 to seat number 1 first and then work on the others may give the same result if we put student number 2 to seat number 2 then put student number 1 to seat number 1 next. The ordering of seating which student first shouldn't matter.