EE 503

Final Part 2 Solution

Fall 2019, 55 Minutes, 100 Points

Problem 6. (20 points.)

a. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence each with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x \leq 1,\ 0 < \theta < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Find the MLE of θ and show its variance goes to 0 as $n \to \infty$.

Solution: The likelihood function is

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}.$$

Taking logs we get

$$\log L(\theta|x) = n\log\theta + (\theta-1)\sum \log(x_i)$$

Taking the derivative of the log likelihood and equating to 0 we get

$$\frac{d}{d\theta}\log L(\theta|x) = \frac{n}{\theta} + \sum \log(x_i) = 0$$

which yields the MLE of θ

$$\hat{\theta} = -\frac{n}{\sum \log(X_i)}.$$

The MLE achieves the CRLB and this distribution is an exponential family so we compute

$$-\frac{d^2}{d\theta^2}\log L(\theta|x) = \frac{n}{\theta^2}.$$

Then with

$$W(X) = -\frac{n}{\sum \log(X_i)}$$

we have the CRLB as

$$\begin{aligned} CRLB &= \frac{(E[W(X)])^2\theta^2}{-E\left[\frac{d^2}{d\theta^2}\log L(\theta|x)\right]} \\ &= \frac{(E[W(X)])^2\theta^2}{n} \end{aligned}$$

which clearly tends to 0 as $n \to \infty$.

b. An information source generates i.i.d. bits X_n for which $P(X_n = 0) = P(X_n = 1) = 1/2$. Suppose X_n is transmitted over K consecutive identical and independent BSCs. This sequence of channel outputs forms a Markov chain. Find both the one-step and two-step transition matrices that relate the input bits from the source to the output bits of the Kth channel, for $K = 1, 2$. You can compute the two-step transition matrix (P^2) with an easy direct calculation (you do not need to compute eigenvalues and eigenvectors).

Solution: Let p denote the probability of bit error. The one-step transition probability matrix is

$$P = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}.$$

The two-step transition probability matrix is

$$P^2 = \begin{bmatrix} 1-2p+2p^2 & 2p-2p^2 \\ 2p-2p^2 & 1-2p+2p^2 \end{bmatrix}.$$

Problem 7. (20 points.) Consider the joint density for random variables X and Y

$$f(x,y) = \begin{cases} (2x+3y)/7, & 0 < x < 2,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the marginal density of X, $f_X(x)$.

Solution:

$$\begin{aligned} f_X(x) &= \int_0^1 f(x,y)dy \\ &= \frac{1}{7}\int_0^1 (2x+3y)dy \end{aligned}$$

which becomes

$$f_X(x) = \begin{cases} \frac{2x}{7} + \frac{3}{14}, & 0 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find the pdf of the random variable $Z = (X+1)^2$.

Solution:

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P\left((X+1)^2 \leq z\right) \\ &= P\left(X \leq \sqrt{z} - 1\right) \\ &= F_X(\sqrt{z} - 1). \end{aligned}$$

Thus,

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= f_x(\sqrt{z}-1)\frac{1}{2\sqrt{z}} \end{aligned}$$

which becomes

$$f_Z(z) = \begin{cases} \frac{1}{7} - \frac{1}{28\sqrt{z}}, & 1 < z < 9, \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 8. (20 points.)

a. Let X be binomially distributed with $n = 400$ and $p = \frac{1}{4}$. Using the central limit theorem find $P(X \leq 108)$. Write your answer using $\Phi(z)$ (the standard normal (Gaussian) cdf).

Solution: Let X_N denote the continuous normal random variable corresponding to the discrete binomial random variable X.

$$\begin{aligned} P(X \leq 108) &\approx P(X_N \leq 108.5) \\ &= P\left(Z \leq \frac{108.5 - E[X]}{\sigma_X}\right), \end{aligned}$$

where Z is standard normal. Now $E[X] = np = 100$ and $\sigma_X = \sqrt{np(1-p)} = \frac{\sqrt{300}}{2}$. Hence,

$$P(X \leq 108) \approx P(Z \leq 0.9815) = \Phi(0.9815)\ [= 0.8368].$$

b. The moment generating function for a random variable X that has a chi squared distribution with 2 degrees of freedom is $M_X(t) = \dfrac{1}{1-2t}$ for $0 \leq t < \frac{1}{2}$. Using this mgf find the mean and variance of X.

Solution: First note that

$$M'_X(t) = \frac{2}{(1-2t)^2}, \quad M''_X(t) = \frac{8}{(1-2t)^3}.$$

The mean of X is then

$$\mu = M'_X(0) = 2$$

and the variance is

$$\sigma^2 = M''_X(0) - \mu^2 = 8 - 4 = 4.$$

Problem 9. (20 points.)

a. Let Y and U be two independent random variables with $Y \sim N(0,1)$ and $P(U = 1) = P(U = -1) = 1/2$. Let $Z = UY$. Show that

i. $Z \sim N(0,1)$.

Solution:

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(UY \le z) = P(UY \le z|U=1)P(U=1) \\ &\quad + P(UY \le z|U=-1)P(U=-1) \\ &= P(Y \le z)\frac{1}{2} + P(Y \ge -z)\frac{1}{2} = F_Y(z)\frac{1}{2} + (1 - F_Y(-z))\frac{1}{2} \end{aligned}$$

so

$$\begin{aligned} f_z(z) &= \frac{d}{dz}F_Z(z) = f_Y(z)\frac{1}{2} - f_Y(-z)(-1)\frac{1}{2} \\ &= f_Y(z) \sim N(0,1) \end{aligned}$$

since $f_Y(z) = f_Y(-z)$.

ii. Y and Z are uncorrelated.

Solution: Note that $E[U] = 0$.

$$E[YZ] = E[UY^2] = E[U][EY^2] = 0 \cdot E[Y^2] = 0 = E[Y]E[Z]$$

so Y and Z are uncorrelated.

iii. Y and Z are **not** independent. Note that uncorrelated implies independent when the random variables are jointly normal but Y and Z are not jointly normal.

Solution: Note that $|Y| = |Z|$ so Y and Z are not independent. Or you could note that

$$P(Z > 1|Y = 1/2) = P(Y > 1|Y = 1/2) = 0$$

so Y and Z are not independent. Observe that $P(Y + Z = 0) = \frac{1}{2}$ (which is not 0) so $Y + Z$ is not normally distributed so Y and Z are not jointly normal.

b. Let $X = (X_1, X_2, \ldots, X_n)$ be i.i.d. where each X_i has distribution function

$$F(x) = \begin{cases} \sqrt{x}, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$. Let $Z_n = n^2 Y_n$. Show Z_n converges in distribution and find the limiting distribution. Note you may use the fact that $\left(1 + \frac{u}{n}\right)^n \to e^u$ as $n \to \infty$.

Solution:

$$P(Y_n \le x) = 1 - P(all\ X_i > x) = 1 - (1 - F(x))^n = (1 - \sqrt{x})^n.$$

$$\begin{aligned} P(Z_n \le x) &= P(n^2 Y_n \le x) = P(Y_n \le x/n^2) \\ &= 1 - \left(1 - \frac{\sqrt{x}}{n}\right)^n \Rightarrow 1 - e^{-\sqrt{x}} \text{ as } n \to \infty. \end{aligned}$$

Problem 10. (20 points.) Let $X_1, X_2, \ldots$ be i.i.d. where each X_i has distribution function F such that $F(x) < 1$ for all x. Let y be a constant and let $Y = \min\{k : X_k > y\}$. Note that Y depends on y, i.e., $Y = Y(y)$.

a. Find $P(Y(y) \leq E[Y(y)])$.

Solution:

$$\begin{aligned} P(Y(y) = k) &= P(X_1 \leq y, X_2 \leq y, \ldots, X_{k-1} \leq y, X_k > y) \\ &= F(y)^{k-1}(1 - F(y)) \end{aligned}$$

so

$$\begin{aligned} P(Y(y) \leq k]) &= \sum_{m=1}^{k-1} F(y)^{m-1}(1 - F(y)) \\ &= (1 - F(y))F(y)^{-1} \left[\frac{F(y) - F(y)^k}{1 - F(y)}\right] \\ &= 1 - F(y)^{k-1}. \end{aligned}$$

Now

$$\begin{aligned} E[Y(y)] &= \sum_{k=1}^{\infty} kF(y)^{k-1}(1 - F(y)) \\ &= (1 - F(y))\frac{d}{dF(y)} \sum_{k=1}^{\infty} F(y)^k \\ &= (1 - F(y))\frac{d}{dF(y)} \frac{F(y)}{1 - F(y)} \\ &= \frac{1}{1 - F(y)}. \end{aligned}$$

Hence,

$$\begin{aligned} P(Y(y) \leq E[Y(y)]) &= 1 - F(y)^{\frac{1}{1-F(y)} - 1} \\ &= 1 - F(y)^{\frac{F(y)}{1-F(y)}}. \end{aligned}$$

b. Find $\lim_{y\to\infty} P(Y(y) \leq E[Y(y)])$.

Solution: Fix y and since we are letting $y \to \infty$ assume y is large here. We can partition the half-open unit interval as

$$[0,1) = \left[0, \frac{1}{y}\right) \cup \left[\frac{1}{y}, \frac{2}{y}\right) \cup \cdots \cup \left[\frac{y-1}{y}, \frac{y}{y}\right).$$

Then, for any $F(y) < 1$ we can find a positive integer m such that

$$\frac{y-m}{y} \leq F(y) < \frac{y-m+1}{y}$$

that is,

$$1 - \frac{m}{y} \leq F(y) < 1 - \frac{m-1}{y}.$$

Since $F(y)$ approaches 1 and $y \to \infty$ then $\frac{m}{y} \to 0$ and therefore $\frac{m-1}{y} \to \frac{m}{y}$ so we may assume that $F(y) \to 1 - \frac{m}{y}$ in our calculations where m depends on y. We then get

$$\frac{F(y)}{1-F(y)} \to \frac{y}{m} - 1.$$

Let $u = \frac{y}{m}$. Then, $F(y) \to 1 - \frac{1}{u}$. Thus,

$$\begin{aligned} P(Y(y) \leq E[Y(y)]) &\to 1 - \left(1 - \frac{1}{u}\right)^{u-1} \\ &\to 1 - \left(1 - \frac{1}{u}\right)^{u} \\ &\to 1 - e^{-1} \text{ as } u \to \infty. \end{aligned}$$