# EE 503
## Final Part 1 Solution
**Fall 2019, 55 Minutes, 100 Points**

**Problem 1.** (20 points.)

a. Let $X$ be a discrete random variable with $P(X=1)=\frac{1}{2}$, $P(X=2)=\frac{1}{4}$ and $P(X=4)=\frac{1}{4}$. Find $E[X]$ and $Var(X)$.

**Solution:**

$$\begin{aligned} E[X] &= 1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+4\cdot\frac{1}{4}=2 \\ E\left[X^2\right] &= 1\cdot\frac{1}{2}+4\cdot\frac{1}{4}+16\cdot\frac{1}{4}=\frac{11}{2}=5.5 \\ Var[X] &= E\left[X^2\right]-(E[X])^2=\frac{3}{2}=1.5. \end{aligned}$$

b. Let $X_n$ be a sequence of i.i.d. Bernoulli (0,1) random variables with $P(X_i=1)=p=1-P(X_i=0)$ and let

$$Y_n = X_1 + X_1X_2 + X_1X_2X_3 + \cdots.$$

Does $Y_n$ converge almost surely to a constant? (Just answer Yes or No – proof is not required.) If it does converge almost surely to a constant indicate the constant. Find the mean of $Y_n$ as $n\to\infty$.

**Solution:** No, $Y_n$ does not converge to a constant almost surely. $Y_n$ is a sum of nonnegative random variables so $Y_n$ is not approaching any constant value. We find (you can assume $0\le p<1$)

$$E[Y_n]=p+p^2+p^3+\cdots=\sum_{k=1}^{\infty}p^k=\frac{p}{1-p},\ \ 0\le p<1.$$

**Problem 2.** (20 points.) A random variable $X$ has pdf

$$f(x) = \begin{cases} \frac{1}{2}x^{-1/2}, & 1 \le x \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the cumulative distribution function (cdf) of $X$.

**Solution:**

$$F(x) = \int_1^x \frac{1}{2}u^{-1/2}du$$

which becomes

$$F(x) = \begin{cases} x^{1/2} - 1, & 1 \le x \le 4, \\ 1, & x > 4, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find $P(X > 2)$. Write the probability you compute in numerical form.

**Solution:**

$$P(X > 2) = 1 - F(2) = 2 - \sqrt{2} = 0.5858.$$

**Problem 3.** (20 points.) Suppose the two-dimensional continuous random variable $(X, Y)$ has joint *pdf*

$$f_{XY}(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find $f_{X|Y}(x|y)$.

**Solution:**

$$\begin{aligned} f_Y(y) &= \int_0^1 f(x,y)dx \\ &= \int_0^1 (x+y)dx \\ &= \frac{1}{2} + y. \end{aligned}$$

Hence,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

which becomes

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{2x+2y}{1+2y}, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

b. Let $B = \{X + Y \geq 1/2\}$. Find $P(B)$.

**Solution:** $P(B) = 1 - P(\bar{B})$, where $\bar{B} = \{X + Y < 1/2\}$.

$$\begin{aligned} P(\bar{B}) &= \int_0^{1/2} \int_0^{1/2-y} f_{XY}(x,y)dxdy \\ &= \int_0^{1/2} \int_0^{1/2-y} (x+y)dxdy \\ &= \frac{1}{24}. \end{aligned}$$

Thus,

$$P(B) = \frac{23}{24}.$$

**Problem 4.** (20 points.) Let $X$ and $Y$ have joint *pdf*

$$f_{XY}(x,y) = \begin{cases} \dfrac{xy}{2}, & 0 \le y \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Z = X + Y$.

a. Find the pdf of $Z$.

**Solution:**

$$F_Z(z) = P(Z \le z) = P(X + Y \le z).$$

Let $W = X$ then $X = W$ and $Y = Z - W$. The transformation has Jacobian 1 thus

$$f_Z(z) = \int f_{XY}(w, z-w)dw.$$

There are two case to consider:

i) $0 \le z \le 2 \Rightarrow \frac{z}{2} \le w \le z$. We get

$$f_Z(z) = \int_{z/2}^{z} \frac{1}{2} w(z-w)dw = \frac{z^3}{24}.$$

ii) $2 < z \le 4 \Rightarrow \frac{z}{2} \le w \le 2$. We get

$$f_Z(z) = \int_{z/2}^{2} \frac{1}{2} w(z-w)dw = -\frac{z^3}{24} + z - \frac{4}{3}.$$

Hence,

$$f_Z(z) = \begin{cases} \dfrac{z^3}{24}, & 0 \le z \le 2, \\ -\dfrac{z^3}{24} + z - \dfrac{4}{3}, & 2 < z \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find $P(Z \le 1)$.

**Solution:**

$$P(Z \le 1) = F_Z(1) = \int_0^1 f_Z(z)dz = \int_0^1 \frac{z^3}{24} dz = \frac{1}{96}.$$

**Problem 5.** (20 points.) The random variable $N$ is Poisson distributed with parameter $\lambda > 0$ with pmf

$$P(N=k) = \begin{cases} \dfrac{\lambda^k}{k!}e^{-\lambda}, & k = 0, 1, 2, \ldots \\ 0, & \text{elsewhere.} \end{cases}$$

Let $p \in (0,1)$ and let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = 1) = p,\ P(X_i = 0) = 1-p$ and let $S_n = X_1 + X_2 + \cdots + X_n$ where we define $S_0 = 0$. Here $N$ and $S_n$ are independent for every $n$. Let $Y = S_N$ (the $N$ in $S_N$ is the Poisson random variable). Let $Z = N - Y$ so that $Y + Z = N$.

a. Find $\lim \dfrac{P(N \geq k)}{P(N = k)}$ where the limit is taken with $\lambda \to \infty,\ k \to \infty,$ $\lambda/k \to a$. Specify any restrictions on $a$ for this limit to exist.

**Solution:** Let $C$ denote the conditions $\lambda \to \infty,\ k \to \infty,\ \lambda/k \to a$. Then

$$\begin{aligned} \lim_C \frac{P(N \geq k)}{P(N=k)} &= \lim_C \frac{\sum_{m=k}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda}}{\frac{\lambda^k}{k!} e^{-\lambda}} = \lim_C \sum_{m=k}^{\infty} \frac{k!}{m!} \lambda^{m-k} \\ &= \sum_{u=0}^{\infty} \frac{k!}{(u+k)!} \lambda^u \end{aligned}$$

which becomes

$$\lim_C \frac{P(N \geq k)}{P(N=k)} = 1 + \frac{\lambda}{k+1} + \frac{\lambda^2}{(k+2)(k+1)} + \frac{\lambda^3}{(k+3)(k+2)(k+1)} + \cdots$$

or

$$\begin{aligned} \lim_C \frac{P(N \geq k)}{P(N=k)} &= 1 + \frac{\lambda}{k(1+1/k)} + \frac{\lambda^2}{k^2(1+2/k)(1+1/k)} \\ &\quad + \frac{\lambda^3}{k^3(1+3/k)(1+2/k)(1+1/k)} + \cdots \\ &= \sum_{u=0}^{\infty} \frac{\lambda^u}{\prod_{i=1}^{u}(k+i)} \\ &\quad + \frac{\lambda^3}{k^3(1+3/k)(1+2/k)(1+1/k)} + \cdots \\ &= \sum_{u=0}^{\infty} \frac{(\lambda/k)^u}{\prod_{i=1}^{u}(1+i/k)} \end{aligned}$$

or

$$\lim_C \frac{P(N \geq k)}{P(N = k)} = \sum_{u=0}^{\infty} a^u$$
$$= 1 + a + a^2 + a^3 + \cdots.$$

Thus,

$$\lim_C \frac{P(N \geq k)}{P(N = k)} = \frac{1}{1-a}$$

or

$$\lim_C \frac{P(N \geq k)}{P(N = k)} = \frac{1}{1-\lambda/k}$$

provided $0 < a = \lambda/k < 1$.

b. Show that $Y$ and $Z$ are independent.

**Solution:**

$$\begin{aligned}
P(Y = y, Z = z) &= P(Y = y, Z = z | N = y + z) P(N = y + z) \\
&= \binom{y+z}{y} p^y q^z \frac{\lambda^{y+z}}{(y+z)!} e^{-\lambda} \\
&= \frac{(\lambda p)^y (\lambda q)^z}{y! z!} e^{-\lambda}.
\end{aligned}$$

$$\begin{aligned}
P(Y = y) &= \sum_{n \geq y} P(Y = y | N = n) P(N = n) \\
&= \sum_{n \geq y} \binom{n}{y} p^y q^{n-y} \frac{\lambda^n}{n!} e^{-\lambda} \\
&= p^y q^{-y} e^{-\lambda} \sum_{n \geq y} \binom{n}{y} \frac{(\lambda q)^n}{n!} \\
&= \frac{p^y e^{-\lambda}}{y!} \sum_{n \geq y} \frac{\lambda^n q^{n-y}}{(n-y)!} \\
&= \frac{p^y e^{-\lambda}}{y!} \lambda^y \sum_{n \geq 0} \frac{\lambda^n q^n}{n!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{n\geq 0} \frac{(\lambda q)^n}{n!} \\
&= \frac{(\lambda p)^y e^{-\lambda(p+q)}}{y!} e^{\lambda q} \\
&= \frac{(\lambda p)^y}{y!} e^{-\lambda p}.
\end{aligned}
$$

$$
\begin{aligned}
P(Z = z) &= \sum_{n\geq y} P(Z = z|N = n)P(N = n) \\
&= \sum_{z\geq 0} P(Y = N - z|N = n)P(N = n) \\
&= \sum_{n\geq z} \binom{n}{n-z} p^{n-z} q^z \frac{\lambda^n}{n!} e^{-\lambda} \\
&= p^{-z} q^z e^{-\lambda} \sum_{n\geq z} \binom{n}{n-z} \frac{(\lambda p)^n}{n!} \\
&= p^{-z} q^z e^{-\lambda} \sum_{n\geq z} \frac{(\lambda p)^n}{(n-z)!z!} \\
&= \frac{p^{-z} q^z e^{-\lambda}}{z!} \sum_{n\geq z} \frac{(\lambda p)^n}{(n-z)!} \\
&= \frac{(\lambda q)^z e^{-\lambda}}{z!} \sum_{u\geq 0} \frac{(\lambda p)^u}{u!} \\
&= \frac{(\lambda q)^z e^{-\lambda(p+q)}}{z!} e^{\lambda p} \\
&= \frac{(\lambda q)^z}{z!} e^{-\lambda q}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P(Y = y)P(Z = z) &= \frac{(\lambda p)^y}{y!} e^{-\lambda p} \frac{(\lambda q)^z}{z!} e^{-\lambda q} \\
&= \frac{(\lambda p)^y (\lambda q)^z}{y!z!} e^{-\lambda} \\
&= P(Y = y, Z = z)
\end{aligned}
$$

so, $Y$ and $Z$ are independent.