EE 464

Spring 2003

Lecture Notes Part 9g

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9.7 Characteristic Functions

Definition: Let X be any random variable. The *characteristic function* (*cf*) of X is given by

$$\Phi_X(\omega) = E\left(e^{i\omega X}\right).$$

Now

$$\Phi_X : \mathbf{R} \mapsto \mathbf{C}$$

by the rule

$$\Phi_X(\omega) = E(\cos\omega X + i\sin\omega X) = E(\cos\omega X) + i(\sin\omega X).$$

$\Phi_X(\omega)$ is defined $\forall\ \omega \in \mathbf{R}$.

For X discrete, the *cf* of X is

$$\Phi_X(\omega) = \sum_k e^{i\omega x_k} P(X = x_k).$$

For X continuous, the *cf* of X is

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)dx.$$

Note: The *cf* in the continuous case is related to the Fourier transform. In fact, we can use the inversion formula of the Fourier transform to conclude

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(\omega)e^{-i\omega x}d\omega.$$

One can also relate the moments of a random variable (when they exist) to its *cf* as was done for the *mgf*.

9.8 Special Moment Functions

Definition: If X is a random variable taking integer values, then its *moment function* is given by

$$\Gamma(z) = E\left(z^X\right) = \sum_i p_i z^i$$

where $p_i = P(X = i)$.

We compute

$$\Gamma'(z) = \frac{d}{dz}\left(\sum_i p_i z^i\right) = \sum_i i p_i z^{i-1}.$$

So

$$\Gamma'(1) = \sum_i i p_i = E(X).$$

Note: Changing the order of differentiation and summation (as was done above) is okay as long as $|z| <$ radius of convergence for $\sum_i p_i z^i$.

If we continue to differentiate we get

$$\Gamma^{(k)} = E\left[X(X-1)\cdots(X-k+1)\right].$$

Special Case: If X is discrete taking values $0, 1, 2, \ldots$ the *probability generating function* of X is the function

$$G_X(s) = E\left(s^X\right), \;\; s \in \mathbf{R},$$

or

$$G_X(s) = \sum_{i=0}^{\infty} s^i P(X = i).$$

This function is used extensively in characterizing random walks and branching processes.

9.9 Applications of Characteristic Functions

In addition to providing a moment theorem as with the *mgf*, the *cf* can aid us in finding the density function of $Y = g(X)$.

Recall

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)dx = E\left(e^{i\omega X}\right).$$

Let $Y = g(X)$. Then,

$$\Phi_Y(\omega) = \int_{-\infty}^{\infty} e^{i\omega y} f_Y(y)dy = E\left(e^{i\omega Y}\right) = E\left(e^{i\omega g(X)}\right).$$

So

$$\Phi_Y(\omega) = \int_{-\infty}^{\infty} e^{i\omega g(x)} f_X(x)dx.$$

If we can write

$$\int_{-\infty}^{\infty} e^{i\omega g(x)} f_X(x)dx$$

as

$$\int_{-\infty}^{\infty} e^{i\omega y} h(y) dy$$

then $f_Y(y) = h(y)$.

Example: Suppose $X \sim N\left(0, \sigma^2\right)$. Let $Y = \alpha X^2,\ \alpha \in \mathbf{R},\ \alpha \neq 0$. Then

$$\Phi_Y(\omega) = \int_{-\infty}^{\infty} e^{i\omega\alpha x^2} f_X(x) dx = \int_{-\infty}^{\infty} e^{i\omega\alpha x^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

$$= \frac{2}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} e^{i\omega\alpha x^2} e^{-x^2/2\sigma^2} dx.$$

Let $y = \alpha x^2,\ dy = 2\alpha x dx = 2\sqrt{\alpha y} dx$. Then

$$\Phi_Y(\omega) = \frac{2}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} e^{i\omega y} e^{-y/2\alpha\sigma^2} \frac{1}{2\sqrt{\alpha y}} dy$$

or

$$\Phi_Y(\omega) = \int_{-\infty}^{\infty} e^{i\omega y} \frac{1}{\sigma\sqrt{2\pi\alpha y}} e^{-y/2\alpha\sigma^2} U(y) dy$$

where,

$$U(y) = \begin{cases} 1, & y \geq 0, \\ 0, & \text{else.} \end{cases}$$

Thus,

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi\alpha y}} e^{-y/2\alpha\sigma^2} U(y).$$