EE 464

Spring 2003

Lecture Notes Part 9f

Christopher Wayne Walker, Ph.D.

9.6 Moment Generating Function

The moment generating function has many uses, one of which is to calculate moments of a random variable.

Definition: Let X be a random variable. The *moment generating function* (*mgf*) of X is given by

$$M_X(s) = M(s) = E\left(e^{sX}\right).$$

For X discrete, the *mgf* of X is

$$M_X(s) = \sum_i e^{sx_i} P(X = x_i).$$

For X continuous, the *mgf* of X is

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx.$$

Recall,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

which converges for all constants x. So,

$$e^{sx} = 1 + sx + \frac{(sx)^2}{2!} + \frac{(sx)^3}{3!} + \cdots.$$

Now

$$M_X(s) = E\left(e^{sX}\right) = E\left(1 + sX + \frac{(sX)^2}{2!} + \frac{(sX)^3}{3!} + \cdots\right).$$

If we assume the *mgf* exists then the expectation of the sum is the sum of the expectations, so

$$M_X(s) = 1 + sE(X) + \frac{s^2 E(X^2)}{2!} + \frac{s^3 E(X^3)}{3!} + \cdots.$$

We can also calculate $M'_X(s)$ by taking the derivative of each term to get

$$M'_X(s) = E(X) + sE(X^2) + \frac{s^2 E(X^3)}{2!} + \cdots.$$

We set $s = 0$ to conclude $M'_X(0) = E(X)$. Also,

$$M''_X(s) = E(X^2) + sE(X^3) + \frac{s^2 E(X^4)}{2!} + \cdots.$$

We see that $M''_X(0) = E(X^2)$. Continuing on leads to the following theorem.

Theorem: $M_X^{(n)}(0) = E(X^n)$.

Note: The *mgf* in the continuous case is related to the Laplace transform.

9.6.1 Examples

Binomial Case: Say X is binomially distributed with parameters n, p. We have

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \ k = 1, 2, 3, \ldots, n.$$

Then

$$\begin{aligned} M_X(s) = E\left(e^{sX}\right) &= \sum_{k=0}^{n} e^{sk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^s)^k (1-p)^{n-k} \\ &= [pe^s + (1-p)]^n . \end{aligned}$$

Thus,

$$M_X(0) = [p + 1 - p]^n = 1 = E\left(X^0\right) = E(1).$$

$$M'_X(s) = n [pe^s + (1-p)]^{n-1} pe^s.$$

$$M'_X(0) = np = E(X).$$

$$\begin{aligned} M''_X(s) &= n [pe^s + (1-p)]^{n-1} pe^s + pe^s n(n-1) [pe^s + (1-p)]^{n-2} pe^s \\ &= np \left[(pe^s + (1-p))^{n-1} e^s + e^s n(n-1) (pe^s + (1-p))^{n-2} pe^s\right]. \end{aligned}$$

$$M''_X(0) = np[1 + (n-1)p] = E\left(X^2\right).$$

So

$$Var(X) = E\left(X^2\right) - [E(X)]^2 = np[1 + (n-1)p] - (np)^2 = np(1-p).$$

Normal Case: Here $X \sim N\left(\mu, \sigma^2\right)$.

$$M_X(s) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{sx} e^{-(x-\mu)^2/2\sigma^2} dx.$$

Let $w = \dfrac{x-\mu}{\sigma} \Rightarrow x = \sigma w + \mu,\ dx = \sigma dw$. Then

$$\begin{aligned}
M_X(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{s(\sigma w + \mu)} e^{-w^2/2} dw \\
&= e^{s\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(w^2 - 2\sigma s w\right)} dw \\
&= e^{s\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left((w-\sigma s)^2 - \sigma^2 s^2\right)} dw \\
&= e^{s\mu + \sigma^2 s^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w-\sigma s)^2} dw.
\end{aligned}$$

Let $v = w - \sigma s,\ dv = dw$ to get

$$M_X(s) = e^{s\mu + \sigma^2 s^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2} dv.$$

But,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2} dv = 1$$

since it is a density function for a mean zero, unit variance random variable (standard normal). Thus,

$$M_X(s) = e^{s\mu + \sigma^2 s^2/2}.$$

Note: For the normal case we have

$$M_X'(s) = \left(\mu + \sigma^2 s\right) e^{s\mu + \sigma^2 s^2/2}$$

and

$$M_X'(0) = \mu = E(X).$$

The *mgf* for a random variable may not exist for any s since some random variables such as the Cauchy do not have finite moments. X is Cauchy if it has *pdf*

$$f(x) = \frac{\alpha}{\pi\left(\alpha^2 + x^2\right)},\ x \in \mathbf{R},\ \alpha > 0.$$

However, the characterisitc function for a random variable always exists.