EE 464

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Lecture Notes Part 9e

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An Interpretation of Expectation:

Suppose we measure the squared distance between a random variable X and a constant b by $(X-b)^2$. Let us find b that minimizes $E[(X-b)^2]$, which gives us a predictor of X. (We do not try to find a b that minimizes $(X-b)^2$ since such a b would depend on X so we could not use it as a predictor.)

Consider

$$E[(X-b)^2] = \int_{-\infty}^{\infty} (x-b)^2 f_X(x)dx.$$

Set

$$\frac{d}{db}E[(X-b)^2] = 0 \Rightarrow \frac{d}{db}\int_{-\infty}^{\infty} (x-b)^2 f_X(x)dx = 0.$$

We can solve this if we can exchange the order of differentiation and integration (justification, in general, requires measure theory concepts). Assuming okay, we get

$$\int_{-\infty}^{\infty} -2(x-b)f_X(x)dx = 0 \Rightarrow \int_{-\infty}^{\infty} x f_X(x)dx = b\int_{-\infty}^{\infty} f_X(x)dx.$$

But

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

thus

$$b = \int_{-\infty}^{\infty} x f_X(x)dx = E(X).$$

We can get this same result another way as follows.

$$E\left[(X-b)^2\right] = E\left[(X-E(X)+E(X)-b)^2\right]$$

$$= E\left[((X-E(X))+(E(X)-b))^2\right]$$

$$= E\left[(X-E(X))^2\right] + 2E\left[(X-E(X))(E(X)-b)\right] + E\left[(E(X)-b)^2\right].$$

Now $(E(X)-b)$ is a constant so

$$E\left[(X-E(X))(E(X)-b)\right] = (E(X)-b)E\left[(X-E(X))\right]$$

$$= (E(X)-b)(E(X)-E(X)) = 0$$

so

$$E\left[(X-b)^2\right] = E\left[(X-E(X))^2\right] + (E(X)-b)^2.$$

We have no control over $E\left[(X-E(X))^2\right]$ since there is no b in this expression. Thus, $E\left[(X-b)^2\right]$ is minimized if we minimize $(E(X)-b)^2$. Since $(E(X)-b)^2 \geq 0$ this term is minimized if $b = E(X)$. Hence,

$$\min_b E\left[(X-b)^2\right] = E\left[(X-E(X))^2\right].$$

9.5 Moments

Definitions: For $k = 1, 2, 3, \ldots$, the *kth moment* of X is

$$m_k = E\left[X^k\right]$$

and the *kth central moment* of X is

$$\mu_k = E\left[(X-E(X))^k\right].$$

Note: The 2nd central moment of X is the variance of X, $Var(X) = \sigma_X^2$. The *standard deviation* of X is $\sigma_X = \sqrt{\sigma_X^2}$.

Normal Case: Consider the mean-zero normal density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.$$

Claim: For $n \geq 1$,

$$E\left[X^n\right] = \begin{cases} 0, & n \text{ odd}, \\ 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n, & n \text{ even}. \end{cases}$$

Proof: If n is odd it is obvious. So assume n is even.

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = 1 \Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sqrt{2\pi}\sigma.$$

Let $\alpha = \dfrac{1}{2\sigma^2}$. We get

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}.$$

Take the derivative with respect to α to get

$$\int_{-\infty}^{\infty} -x^2 e^{-\alpha x^2} dx = -\frac{1}{2}\sqrt{\pi}\alpha^{-3/2}.$$

Cancel the minus signs and continue taking derivatives to get upon the kth derivative

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k} \frac{\sqrt{\pi}}{\sqrt{\alpha^{2k+1}}}.$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k} \sqrt{\pi (2\sigma^2)^{2k+1}}.$$

Using $n = 2k$ we deduce

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^n e^{-x^2/2\sigma^2} dx = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n, \; n \text{ even}.$$

The left hand side of the above result is $E[X^n]$.

Sometimes it is useful to bound probabilities, especially when the probabilities are difficult to calculate or the density and/or the distribution functions are not even known. The Tchebycheff (Chebyshev) Inequality helps us here.

Theorem: (*Tchebycheff Inequality*). For any $\epsilon > 0$,

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2},$$

where $\mu = E(X)$ and $\sigma^2 = Var(X)$.

Proof:

$$P(|X - \mu| \geq \epsilon) = \int_{-\infty}^{-\mu-\epsilon} f(x)dx + \int_{\mu+\epsilon}^{\infty} f(x)dx = \int_{|X-\mu| \geq \epsilon} f(x)dx.$$

Now

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \geq \int_{|X-\mu|\geq\epsilon} (x-\mu)^2 f(x)dx \geq \epsilon^2 \int_{|X-\mu|\geq\epsilon} f(x)dx.$$

But,

$$\int_{|X-\mu|\geq\epsilon} f(x)dx = P(|X - \mu| \geq \epsilon).$$

So

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$