EE 464

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Lecture Notes Part 9c

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Inverse Problem:

i. Given a random variable X with distribution function $F_X(x)$, find $g(x_0)$ so that $U = g(X)$ is uniform in (0, 1).

Claim $g(x_0) = F_X(x_0)$ works.

Proof:

$$F_U(u_0) = P(U \leq u_0) = P(X \leq x_0) = F_X(x_0) = u_0.$$

ii. Given a random variable $U \sim U(0,1)$, find $g(u_0)$ so that $Y = g(U)$ has some desired distribution function $F_Y(y_0)$.

Claim $g(u_0) = F_Y^{-1}(u_0)$ works.

Proof:

$$Y = F_Y^{-1}(U) \Leftrightarrow P(Y \leq y_0) = P(F_Y^{-1}(U) \leq y_0) = P\left(U \leq F_Y(y_0)\right) = F_Y(y_0).$$

iii. Given X with distribution function $F_X(x_0)$, find $g(x_0)$ so that $Y = g(X)$ has some desired distribution function $F_Y(y_0)$.

Claim $g(x_0) = F_Y^{-1}(F_X(x_0))$ works.

Proof:

$$Y = F_Y^{-1}(F_X(X)) \Leftrightarrow P(Y \leq y_0) = P(F_Y^{-1}(F_X(X)) \leq y_0) = P\left(F_X(X) \leq F_Y(y_0)\right)$$

$$= F_Y(y_0) \text{ since } F_X(X) \text{ is } U(0,1) \text{ from (i) above.}$$

Example: Say $X \sim U(0,1)$. We desire a distribution function

$$F_Y(y_0) = \begin{cases} 1 - e^{-y_0}, & y_0 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $y_0 = g(x_0)$.

Solution: Using (ii) above we let $g(x_0) = F_Y^{-1}(x_0)$. We then solve $y_0 = F_Y^{-1}(x_0)$

to get $x_0 = F_Y(y_0)$, so set $x_0 = 1 - e^{-y_0}$. Hence, $y_0 = -\ln(1 - x_0)$ and thus $g(x_0) = -\ln(1 - x_0)$.

9.2 Expectations

Often we would like to know the mean (or average or expected value) of a random variable resulting from a random experiment.

9.2.1 Discrete Case

Say X has values in $\{x_1, x_2, x_3, \ldots\}$ and $P(X = x_i) = f(x_i) = p_i$.

Definition: The *expected value* of X is given by

$$E(X) = \sum_i x_i f(x_i)$$

whenever this sum exists.

Recall $f(x_i) \geq 0$ and $\sum_i f(x_i) = 1$. So, $E(X)$ is the average of the values of X with each value weighted according to its probability of occurrence.

Lemma: $E[g(X)] = \sum_i g(x_i) f_X(x_i)$ where $f_X(x_i) = P(X = x_i) = p_i$.

Proof: Let $Y = g(X)$, then $E[g(X)] = E(Y) = \sum_i y_i f_Y(y_i)$. Now

$$\sum_i g(x_i) f_X(x_i) = \sum_i \left(\sum_j g(x_j) f_X(x_j) \right)$$

where the inner sum is over all indices j for which $g(x_j) = y_i$, for some fixed y_i. Thus, all the terms $g(x_j)$ are constant in the inner sum. Hence,

$$\sum_i g(x_i) f_X(x_i) = \sum_i y_i \sum_j f_X(x_j).$$

But,

$$\sum_j f_X(x_j) = \sum_j P(X = x_j) = P(Y = y_i) = f_Y(y_i).$$

So,

$$\sum_i g(x_i) f_X(x_i) = \sum_i y_i f_Y(y_i).$$

Theorem:

a. $X \geq 0 \Rightarrow E(X) \geq 0$.

b. $E(aX + bY) = aE(X) + bE(Y)$.

c. $E(1) = 1$.

Proof: Easy. Left as an exercise.

Note: Part (a) of this theorem implies expectation is a linear operator.

9.2.2 Continuous Case

Definition: The *expected value* of X is given by

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

whenever this integral exists.

Example: Say X has *pdf*

$$f_X(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$E(X) = \int_0^1 x(3x^2) dx = 3/4.$$

Lemma: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$.

Proof: Omitted.

Theorem:

a. $X \geq 0 \Rightarrow E(X) \geq 0$.

b. $E(aX + bY) = aE(X) + bE(Y)$.

c. $E(1) = 1$.

Proof: Easy. Left as an exercise.