EE 464

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Lecture Notes Part 9b

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9.1.2 Continuous Case

Here X is a continuous random variable and g is a continuous function. So, $Y = g(X)$ is a continuous random variable. We seek the *pdf* of Y, i.e., $f_Y(y)$.

General Procedure:

i. Obtain $F_Y(y) = P(Y \leq y)$ by finding the event A (in the range space of X) which is equivalent to the event $\{Y \leq y\}$.

ii. Differentiate $F_Y(y)$ to get $f_Y(y)$.

iii. Determine those values in the range space of Y for which $f_Y(y) > 0$.

Example: Let

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $g(x) = 3x + 1$. Then

$$F_Y(y) = P(Y \leq y) = P(3X + 1 \leq y) = P\left(X \leq \frac{y-1}{3}\right)$$

$$= \int_0^{\frac{y-1}{3}} 2x dx = \left(\frac{y-1}{3}\right)^2.$$

Thus,

$$f_Y(y) = F_Y'(y) = \frac{2}{9}(y-1).$$

Now $f_X(x) > 0$ for $0 < x < 1$, therefore $f_Y(y) > 0$ for $1 < y < 4$.

There is another way of getting the same result. Consider

$$F_Y(y) = P(Y \leq y) = P(3X + 1 \leq y) = P\left(X \leq \frac{y-1}{3}\right) = F_X\left(\frac{y-1}{3}\right).$$

Then,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-1}{3}\right) = \frac{dF_Y(y)}{du}\frac{du}{dy} = \frac{dF_X(u)}{du}\frac{du}{dy}$$

where $u = (y-1)/3$. So,

$$f_Y(y) = F'_X(u)\frac{du}{dy} = f_X(u)\frac{du}{dy} = 2\left(\frac{y-1}{3}\right)\frac{1}{3} = \frac{2}{9}(y-1).$$

The following theorem is very useful if the conditions of the theorem are met.

Theorem: Let X be a continuous random variable with *pdf* $f_X(x) > 0$ for $a < x < b$. Suppose that $y = g(x)$ is a strictly monotone (strictly increasing or strictly decreasing) function of x. Assume that g is differentiable (and hence continuous) for all x. Then $Y = g(X)$ has *pdf*

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

where x is expressed in terms of y, i.e., $x = g^{-1}(y)$. Hence.

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right|.$$

If g is increasing then g is nonzero for those values of y satisfying $g(a) < y < g(b)$. If g is decreasing then g is nonzero for y satisfying $g(b) < y < g(a)$.

Proof: First assume g is a strictly increasing function. Then

$$F_Y(y) = P(Y \le y) = P\left(g(X) \le y\right) = P\left(X \le g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right).$$

Thus,

$$f_Y(y) = \frac{dF_Y(x)}{dx}\frac{dx}{dy} = \frac{dF_X(x)}{dx}\frac{dx}{dy}$$

where $x = g^{-1}(y)$. Hence,

$$f_Y(y) = f_X(x)\frac{dx}{dy} = f_X(x)\left|\frac{dx}{dy}\right|.$$

Now assume g is a strictly decreasing function. Then,

$$F_Y(y) = P(Y \le y) = P\left(g(X) \le y\right) = P\left(X \ge g^{-1}(y)\right)$$

$$= 1 - P\left(X < g^{-1}(y)\right) = 1 - F_X\left(g^{-1}(y)\right).$$

So,

$$\frac{dF_Y(y)}{dy} = \frac{dF_Y(y)}{dx}\frac{dx}{dy}, \ x = g^{-1}(y)$$

or

$$f_Y(y) = \frac{d}{dx}\left[1 - F_X(x)\right]\frac{dx}{dy} = -f_X(x)\frac{dx}{dy}.$$

But, $\frac{dx}{dy} < 0$ since $g(x)$ is strictly decreasing. Hence,

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|.$$

Note: If $y = g(x)$ is not a strictly monotone function of x, we cannot apply the above theorem directly but we can still use the general method.

Example: Suppose

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1, \\ \\ 0, & \text{elsewhere.} \end{cases}$$

Let $g(x) = x^2$. This function is not monotone over the interval (-1, 1). Here

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) + P(X = -\sqrt{y}).$$

Thus,

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} - \frac{f_X(-\sqrt{y})}{-2\sqrt{y}} = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

or

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2\sqrt{y}}, \quad 0 < y < 1.$$

The result obtained in this example gives the following theorem.

Theorem: Let X be a continuous random variable with *pdf* $f_X(x)$. Let $Y = X^2$. Then the *pdf* of Y is

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right].$$

Example: Suppose

$$f_X(x) = \begin{cases} 3x^2, & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = e^{-X} = g(X)$. Note that $g(x)$ is monotone in $[0, 1]$. Thus,

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

$$g(x) = e^{-x} \text{ or } y = e^{-x} \Rightarrow -x = \ln y \Rightarrow x = -\ln y = g^{-1}(y).$$

Thus,

$$f_Y(y) = 3(-\ln y)^2 \left| \frac{-1}{y} \right| = 3(\ln y)^2 \frac{1}{y}.$$

Endpoints: $x = 0 \Rightarrow y = 1, \quad x = 1 \Rightarrow y = e^{-1}$. So,

$$f_Y(y) = \begin{cases} 3(\ln y)^2 \frac{1}{y}, & e^{-1} \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$