EE 464

Spring 2003

Lecture Notes Part 9a

Christopher Wayne Walker, Ph.D.

9.0 Functions of One Random Variable

Often we are concerned with functions of a random variable [recall a random variable itself is a function of the outcome of an experiment]. If $y = g(x)$ is a real-valued function then $Y = g(X)$ is a random variable. Given $f_X(x)$ or $F_X(x)$ we seek $f_Y(y)$ and $F_Y(y)$.

Definition: Let C be an event (some subset) associated with the range space of Y, R_Y. Define $B \subset R_X$ as

$$B = \{x \in R_X : g(x) \in C\}.$$

Then, B and C are called *equivalent events* (B occurs if and only if C occurs).

Definition: Let X be a random variable defined on the sample space Ω. Let R_X be the range space of X. Let g be a real-valued function and compute the random variable $Y = g(X)$ with range space R_Y. For any $C \subset R_Y$ define

$$P(C) = P\left(\{x \in R_X : g(x) \in C\}\right).$$

Note

$$P(C) = P\left(\{\omega \in \Omega : g\left(X(\omega)\right) \in C\}\right).$$

Example: Let X be a continuous random variable with *pdf*

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $g(x) = 2x + 1$. Then $R_X = \{x : x > 0\}$ while $R_Y = \{y : y > 1\}$. Define the event C as $C = \{Y \geq 5\}$, i.e., $C = \{\omega : g\left(X(\omega)\right) \geq 5\} = \{\omega : Y(\omega) \geq 5\}$. Now $y \geq 5$ iff $2x + 1 \geq 5$ iff $x \geq 2$. So C is equivalent to $B = \{X \geq 2\}$. Now

$$P(X \geq 2) = \int_2^\infty e^{-x} dx = e^{-2}$$

so

$$P(Y \geq 5) = e^{-2}.$$

9.1 Finding the Distribution of $g(X)$

9.1.1 Discrete Case

General Procedure:

1) First we consider the case where X is discrete and Y is discrete.

Let $x_{i1}, x_{i2}, \ldots$ represent the X-values having the property $g(x_{ij}) = y_i, \ \forall\ j$. Then

$$\begin{aligned} f_Y(y_i) = P(Y = y_i) &= P(X = x_{i1}) + P(X = x_{i2}) + \cdots \\ &= f_X(x_{i1}) + f_X(x_{i2}) + \cdots \end{aligned}$$

i.e., to evaluate the probability of the event $\{Y = y_i\}$, find the equivalent event in terms of X and add all the corresponding probabilities together.

Example: Let X have possible values $1, 2, 3, \ldots$ and

$$P(X = n) = (1/2)^n, \ n = 1, 2, \ldots.$$

Let

$$Y = \begin{cases} 1, & x \text{ is even} \\ -1, & x \text{ is odd.} \end{cases}$$

$$\begin{aligned} P(Y = 1) &= P(X = 2) + P(X = 4) + \cdots = (1/2)^2 + (1/2)^4 + \cdots \\ &= \sum_{i=1}^{\infty} (1/2)^{2i} = \sum_{i=1}^{\infty} (1/4)^i = \frac{1/4}{1 - 1/4} = 1/3 \end{aligned}$$

and

$$P(Y = -1) = 1 - P(Y = 1) = 2/3.$$

2) It may turn out that X is a continuous random variable while Y is discrete. For example, X may assume all real values and $Y = 1$ if $X \geq 0$ and $Y = -1$ if $X < 0$. So $P(Y = 1) = P(X \geq 0)$ and $P(Y = -1) = P(X < 0)$. In general, if $\{Y = y_i\}$ is equivalent to an event, say A, in the range space of X, then

$$f_Y(y_i) = P(Y = y_i) = \int_A f_X(x)dx.$$