EE 464

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Lecture Notes Part 8d

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8.6.2 Total Probability and Bayes' Theorem

Let $A_1, \ldots, A_n$ be a partition of Ω. Then

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n).$$

Let $B = \{X \leq x\}$. Then

$$P(X \leq x) = P(X \leq x|A_1)P(A_1) + \cdots + P(X \leq x|A_n)P(A_n).$$

$$\Rightarrow F(x) = F(x|A_1)P(A_1) + \cdots F(x|A_n)P(A_n)$$

$$\Rightarrow f(x) = f(x|A_1)P(A_1) + \cdots f(x|A_n)P(A_n).$$

Also,

$$P(A|X \leq x) = \frac{P(A, X \leq x)}{P(X \leq x)} = \frac{P(X \leq x|A)P(A)}{P(X \leq x)} = \frac{F(x|A)}{F(x)}P(A).$$

Similarly,

$$P(A|x_1 < X \leq x_2) = \frac{P(A, x_1 < X \leq x_2)}{P(x_1 < X \leq x_2)} = \frac{P(x_1 < X \leq x_2|A)P(A)}{P(x_1 < X \leq x_2)}$$

$$= \frac{F(x_2|A) - F(x_1|A)}{F(x_2) - F(x_1)}P(A).$$

In the above we have conditioned on events like $\{X \leq x\}$ or $\{x_1 < X \leq x_2\}$. We cannot use the above development directly when conditioning on the event $\{X = x\}$ (since this event has zero probability in the continuous case). We define

$$P(A|X = x) = \lim_{\Delta x \to 0} P(A|x < X \leq x + \Delta x).$$

Recall,

$$P(A|x_1 < X \leq x_2) = \frac{F(x_2|A) - F(x_1|A)}{F(x_2) - F(x_1)}P(A).$$

Let $x_1 = x,\ x_2 = x + \Delta x$. Then

$$P(A|x < X \leq x + \Delta x) = \frac{F(x + \Delta x|A) - F(x|A)}{F(x + \Delta x) - F(x)}P(A)$$

$$= \frac{\frac{F(x + \Delta x|A) - F(x|A)}{\Delta x}}{\frac{F(x + \Delta x) - F(x)}{\Delta x}}\ P(A).$$

Now let $\Delta x \to 0$ to get

$$P(A|X=x) = \frac{f(x|A)}{f(x)} P(A)$$

$$\Rightarrow \int_{-\infty}^{\infty} P(A|X=x) f(x) dx = P(A) \int_{-\infty}^{\infty} f(x|A) dx = P(A)$$

or

$$P(A) = \int_{-\infty}^{\infty} P(A|X=x) f(x) dx.$$

This is the continuous version of the total probability theorem. Compare to

$$P(A) = \sum_{i=1}^{n} P(A|B_i) P(B_i)$$

where $B_1, \ldots, B_n$ form a partition of Ω.

Recall,

$$P(A|X=x) = \frac{f(x|A)}{f(x)} P(A).$$

Then

$$f(x|A) = \frac{P(A|X=x) f(x)}{P(A)}.$$

Using the above expression for $P(A)$ we get

$$f(x|A) = \frac{P(A|X=x) f(x)}{\int_{-\infty}^{\infty} P(A|X=x) f(x) dx}.$$

This is the continuous version of Bayes' theorem. Compare to

$$P(B_i|A) = \frac{P(A|B_i) P(B_i)}{\sum_{i=1}^{n} P(A|B_i) P(B_i)}$$

where $B_1, \ldots, B_n$ form a partition of Ω.