EE 464

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Lecture Notes Part 8c

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8.6 Conditional Distribution and Density Functions

8.6.1 Definitions and Derivations

Sometimes we wish to know probabilities of certain events associated with the random variable X given knowledge concerning those events.
Example: Roll a fair die. Then

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Let X equal the face value of the die. Then

$$P(X = i) = 1/6, \; i = 1, 2, \ldots, 6.$$

But

$$P(X = 1|X \text{ is even}) = 0, \quad P(X = 2|X \text{ is even}) = 1/3.$$

Let A be the event that $X = 2$. Let B be the event that X is even.

Note: Events are defined as subsets of the sample space to which we can assign probabilities. Since we define a random variable as a deterministic function given the outcome of an experiment we can equivalently relate random variables to events. So, it makes sense to say "A is the event $X = 2$" which really means"A is the event of getting 2 on the roll of a die in which case $X = 2$."

Now

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(X = 2, X \text{ is even})}{P(X \text{ is even})}$$
$$= \frac{P(X = 2)}{P(X \text{ is even})} = \frac{1/6}{1/2} = 1/3.$$

Note: $\{\omega : X(\omega) = 2\} \subset \{\omega : X(\omega) \text{ is even}\}$ so

$$P(\{\omega : X(\omega) = 2\} \cap \{\omega : X(\omega) \text{ is even}\}) = P(\{\omega : X(\omega) = 2\}) = P(X = 2).$$

Definition: The *conditional distribution* $F(x|M)$ of the random variable X, given the event M occurs, is defined as

$$F(x|M) = P(X \leq x|M) = \frac{P(X \leq x, M)}{P(M)}$$

where $P(M) \neq 0$.

Properties:

i. $F(\infty|M) = P(X \leq \infty|M) = 1, \quad F(-\infty|M) = P(X \leq -\infty|M) = 0.$

ii. $P(x_1 < X \leq x_2|M) = P(X \leq x_2|M) - P(X \leq x_1|M)$

$$= F(x_2|M) - F(x_1|M) = \frac{P(X \leq x_2, M)}{P(M)} - \frac{P(X \leq x_1, M)}{P(M)}$$

$$= \frac{P(x_1 < X \leq x_2, M)}{P(M)}.$$

Definition: The *conditional density* $f(x|M)$ is the derivative of $F(x|M)$ with respect to x, i.e.,

$$f(x|M) = \frac{dF(x|M)}{dx} = \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x|M)}{\Delta x}.$$

Special Cases

i. Let M be the event $\{X \leq a\}$ where $P(X \leq a) \neq 0$. Then,

$$F(x|M) = F(x|X \leq a) = P(X \leq x|X \leq a) = \frac{P(X \leq x, X \leq a)}{P(X \leq a)}.$$

If $x \geq a$ then

$$\{X \leq x, X \leq a\} = \{\{X \leq x\} \cap \{X \leq a\}\} = \{X \leq a\}$$

which implies

$$F(x|X \leq a) = \frac{P(X \leq a)}{P(X \leq a)} = 1.$$

If $x < a$ then

$$\{X \leq x, X \leq a\} = \{X \leq x\}$$

which implies

$$F(x|X \leq a) = \frac{P(X \leq x)}{P(X \leq a)} = \frac{F(x)}{F(a)}.$$

Now

$$f(x|X \leq a) = \frac{dF(x|X \leq a)}{dx}$$

so

$$f(x|X \leq a) = \begin{cases} 0, & x \geq a \\ \dfrac{f(x)}{F(a)}, & x < a. \end{cases}$$

ii. Let M be the event $\{b < X \leq a\}$ where $F(a) \neq F(b)$. Then,

$$F(x|b < X \leq a) = \frac{P(X \leq x, b < X \leq a)}{P(b < X \leq a)}.$$

If $x \geq a$ then

$$\{X \leq x, b < X \leq a\} = \{b < X \leq a\}$$

which implies

$$F(x|b < X \leq a) = \frac{F(a) - F(b)}{F(a) - F(b)} = 1.$$

If $b \leq x < a$ then

$$\{X \leq x, b < X \leq a\} = \{b < X \leq x\}$$

which implies

$$F(x|b < X \leq a) = \frac{F(x) - F(b)}{F(a) - F(b)}.$$

If $x < b$ then

$$\{X \leq x, b < X \leq a\} = \emptyset$$

which implies

$$F(x|b < X \leq a) = 0.$$

Thus

$$f(x|b < X \leq a) = \begin{cases} \dfrac{f(x)}{F(a) - F(b)}, & b \leq x < a \\ 0, & else. \end{cases}$$

Examples of conditional distribution calculations will be given in class.