EE 464

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Lecture Notes Part 8b

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8.4 Density Functions

Definition: If X is continuous then

$$f_X(x) = \frac{dF_X(x)}{dx}$$

is called the *probability density function* of X. We may write $f(x) = f_X(x)$ when dealing with just one random variable.

Definition: If X is discrete and $P(X = x_i) = p_i$ then

$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

is called the *probability mass function*.

Here, $f(x_i) = p_i$ and

$$\delta(u) = \begin{cases} 1, & u = 0 \\ 0, & elsewhere. \end{cases}$$

In the continuous case

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du$$

and in the discrete case

$$F_X(x) = \sum_{u:u \leq x} f_X(u).$$

Lemma: Let f be a density function or probability mass function. Then

i. $f(x) \geq 0$.

ii. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$ (continuous case)

or $\displaystyle\sum_i f(x_i) = 1$ (discrete case).

Proof:

i. $F(-\infty) = 0$ and the monotonicity of $F(x)$ implies $f(x) \geq 0$.

ii. $F(+\infty) = 1 \Rightarrow \int_{-\infty}^{\infty} f(u)du = 1$ or $\sum_i f(x_i) = 1$.

8.5 Examples of Random Variables

Normal or Gaussian: This is the most important density function for us. It has the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in \mathbf{R}.$$

It turns out that the mean of X is μ and the variance of X is σ^2. We will say more about this later.

Notation: If the random variable X is normal we write

$$X \sim N(\mu, \sigma^2).$$

The corresponding distribution function is

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \; e^{-(y-\mu)^2/2\sigma^2} dy.$$

Special Case: If $\mu = 0$ and $\sigma^2 = 1$ we have the *standard normal.* Here

$$X \sim N(0, 1)$$

and

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \; e^{-y^2/2} dy.$$

By translation and scaling we can compute the normal distribution function by using the standard normal distribution function. We will see this later.

Note: One can prove that there does not exist a closed form function G such that $G(x) = \int e^{-x^2} dx$, so we cannot integrate the normal density analytically (except when $x \to \infty$, then we can show the integral goes to 1). We must make use of tables or approximation formulas to evaluate the integral.

Uniform: X is uniform between x_1 and x_2 if it has density

$$f(x) = \begin{cases} \frac{1}{x_2 - x_1}, & x_1 \le x < x_2 \\ 0, & elsewhere. \end{cases}$$

Notation: If the random variable X is uniform we write

$$X \sim U(x_1, x_2).$$

Bernoulli: X is Bernoulli if it can take on only two values.

Binomial: X is binomial of order n if

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \ k = 0, 1, 2 \ldots, n$$

where $q = 1 - p$. The mean of X is np (will prove later).

Notation: If the random variable X is binomial we write

$$X \sim B(n, p).$$

Say we repeat an experiment n times, where each time $P(A) = p$ (so $P(\overline{A}) = 1 - p = q$). Then each experiment can be thought of as a Bernoulli trial, i.e., at each trial either A or $\overline{A}$ occurs. Let X be the total number of times A occurs in n trials. In a sequence of trials we can choose the k places to put A in $\binom{n}{k}$ ways. We put $\overline{A}$ in the rest. Thus,

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \ k = 0, 1, 2 \ldots, n.$$

So we can think of the binomial distribution in terms of a sum of Bernoulli trials.

Geometric: X is geometric if

$$P(X = k) = (1 - p)^{k-1} p.$$

Here X can be thought of as the number of trials needed before some event A occurs for the first time.

<u>Poisson:</u> X is Poisson if

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!},\ \ k = 0, 1, 2 \ldots, n$$

for some $\lambda > 0$.