EE 464

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Lecture Notes Part 8a

Christopher Wayne Walker, Ph.D.

8.0 Random Variables

8.1 Definitions and Comments

Definition: For (Ω, F, P), a probability space, a *random variable* is a function $X : \Omega \to \mathbf{R}$ (for each outcome $\omega \in \Omega$ we have a number $X(\omega) \in \mathbf{R}$) with the property that

$$\{\omega : X(\omega) \leq x\} \in F \ \forall x \in \mathbf{R}.$$

Examples:

1. Toss a coin n times. Let X be the number of heads that you observe.
2. Throw a dart at a dartboard. Let X be the score.
3. Wait for a bus. Let X be the waiting time in minutes.

Convention: Random variables are denoted with capital letters and their values are denoted with small letters.

Remarks:

i. $\{\omega : X(\omega) \leq x\} \in F$ by definition.

ii. $\{\omega : X(\omega) < x\} = \bigcup_{n=1}^{\infty} \{\omega : X(\omega) \leq x - 1/n\} \in F$ since $\{\omega : X(\omega) \leq x - 1/n\} \in F$ by definition and the union of such events is in F by the rules of a σ-field.

iii. $\{\omega : X(\omega) = x\} = \{\omega : X(\omega) \leq x\} \setminus \{\omega : X(\omega) < x\} \in F$ since $\{\omega : X(\omega) < x\} \in F$ by (ii), $\{\omega : X(\omega) \leq x\} \in F$ by definition and the set difference of such events is in F by the rules of a σ-field.

iv. $\{\omega : a \leq X(\omega) \leq b\} = \{\omega : X(\omega) \leq b\} \setminus \{\omega : X(\omega) < a\} \in F$ since $\{\omega : X(\omega) < a\} \in F$ by (ii), $\{\omega : X(\omega) \leq b\} \in F$ by definition and the set difference of such events is in F by the rules of a σ-field.

So these (and other sets similar in form) are legitimate events and we can talk about their probabilities.

Note: Often we are more concerned with $P(\{\omega : X(\omega) \in I\})$ for some $I \subset \mathbf{R}$ than we are with (Ω, F, P).

8.2 Distribution Functions

Definition: The *distribution function* of a random variable X is the function

$$F_X = F : \mathbf{R} \to [0,1]$$

given by

$$F(x) = P(\{\omega : X(\omega) \leq x\}) \; \forall x \in \mathbf{R}.$$

Notation: $\{\omega : X(\omega) \leq x\}$ is abbreviated $\{X \leq x\}$. So we write

$$F(x) = P(X \leq x)$$

or

$$F_X(x) = P(X \leq x).$$

A couple of examples we be provided in class.

Lemma: Let F be a distribution function. Then,

i. $F(x) \to 0$ as $x \to -\infty$, $\quad F(x) \to 1$ as $x \to +\infty$.

ii. F is increasing, i.e., $x < y \Rightarrow F(x) \leq F(y)$.

iii. F is right continuous, i.e., $\lim_{h \downarrow 0} F(x+h) = F(x)$.

A proof of this lemma will be provided in class.

8.3 Discrete and Continuous Random Variables

Definition: A random variable X is *discrete* if it takes values in a finite or countably infinite set $\{x_1, x_2, \ldots\}$.

In the discrete case F_X has jumps at the points x_i and is flat between jumps. Here

$$F_X(x_i) = P(X \leq x_i) = \sum_{k=1}^{i} P(X = x_i).$$

Definition: A random variable X is *continuous* if there exists a function

$$f : \mathbf{R} \to [0, \infty)$$

such that

$$F_X(x) = \int_{-\infty}^{x} f(u)du.$$

In this case F_X has no jumps. So F_X continuous implies $P(X = x) = 0 \ \forall x$ and

$$P(a < X \leq b) = \int_{a}^{b} f(u)du.$$

We may have a combination of continuous and discrete distributions. These are called mixtures.

Several examples will be provided in class illustrating distributions.