EE 464

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Lecture Notes Part 3b

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3.2 Probability Space

Let Ω be a sample space with F a σ-field. We wish to assign a number $P(A)$ to each event A representing the likelihood the event will occur.

Consider two motivations for $P(A)$.

i. Ω is finite with all outcomes equally likely. Take

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}.$$

Note: $P(\emptyset) = 0, \quad P(\Omega) = 1, \quad 0 \leq P(A) \leq 1$. If A and B are disjoint, i.e., $A \cap B = \emptyset$, then

$$\begin{aligned} P(A \cup B) &= \frac{\text{number of elements in } A + \text{number of elements in } B}{\text{number of elements in } \Omega} \\ &= P(A) + P(B). \end{aligned}$$

ii. Ω is countably infinite. Repeat the experiment many times. Take

$$P(A) = \lim_{n \to \infty} \frac{\text{number of occurrences of } A \text{ in } n \text{ trials}}{n}.$$

Definition: A *probability measure* P on (Ω, F) is a function

$$P : F \to [0,\ 1]$$

that maps

$$A \mapsto P(A)$$

that satisfies the following axioms of probability:

i. $P(\emptyset) = 0, \quad P(\Omega) = 1, \quad 0 \leq P(A) \leq 1$ (redundant)

ii. If $A_1,\ A_2, \ldots$ is a sequence of pairwise disjoint events then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

.

The triple $(\Omega,\ F,\ P)$ is called a *probability space*.

Remark: If we take $A_3 = A_4 = \cdots = \emptyset$ in (ii) above we get

ii$'$. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.

(ii$'$) is called *finite additivity.* (ii) is called *countable additivity.*

Some consequences of the above will be presented in class.

Examples of Probability Spaces:

i. Ω is finite with all outcomes equally likely. $F = \sigma$-field$=$ all subsets of Ω. Take
$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}.$$

ii. Ω is finite, $\Omega = \{\omega_1,\ \omega_2, \ldots,\ \omega_n\}$, $F = \sigma$-field$=$ all subsets of Ω. Say, $P(\{\omega_i\}) = P(\omega_i) = p_i$, where $p_1,\ p_2, \ldots,\ p_n$ satisfy $p_i \geq 0$ for all i and $p_1 + p_2 + \cdots + p_n = 1$. Take
$$P(A) = \sum_{i:\omega_i \in A} p_i.$$

iii. Ω is countably infinite, $\Omega = \{\omega_1,\ \omega_2, \ldots\}$, $F = \sigma$-field$=$ all subsets of Ω. Say, $P(\{\omega_i\}) = p_i$, where $p_1,\ p_2, \ldots$ satisfy $p_i \geq 0$ for all i and $p_1 + p_2 + \cdots = 1$. Take
$$P(A) = \sum_{i:\omega_i \in A} p_i.$$

iv. $\Omega = [0,\ 1]$. Let $A \subset \Omega$. It is sensible to take
$$P(A) = \frac{\text{length of } A}{\text{length of } \Omega}.$$

This is okay for A consisting of a collection of intervals but not for all subsets of Ω. We can find a "bad" A (will show later).