

EE 464

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Lecture Notes Part 3a

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3.0 Probability Measure

3.1 Sample Space and Events

Definition: The *sample space* Ω is the set of all possible outcomes of a random experiment.

Definition: An *event* is a (particular kind of) subset of the sample space (this will be clarified later by showing that not all subsets are allowed to be events).

Examples:

- i. Choose an integer between 1 and 5. Then

$$\Omega = \{1, 2, 3, 4, 5\}.$$

We see that $\{2, 4\}$ is the event that an even number was chosen.

- ii. Choose a real number in the interval $[0, 1]$. Then

$$\Omega = [0, 1].$$

Note that Ω is uncountable (cannot be put into a 1-1 correspondence with the integers) and not every subset is an event (we will prove this later). But any open or closed subset of $[0, 1]$ is an event, in particular, any subinterval of $[0, 1]$ is an event.

Rules for Events

- i. \emptyset and Ω are events.
- ii. If \mathbf{A} and \mathbf{B} are events then so are $\mathbf{A} \cap \mathbf{B}$, $\mathbf{A} \cup \mathbf{B}$, $\mathbf{B} \setminus \mathbf{A}$ and $\overline{\mathbf{A}} = \Omega \setminus \mathbf{A}$ (note: $b \in \mathbf{B} \setminus \mathbf{A}$ iff $b \in \mathbf{B}$ but $b \notin \mathbf{A}$). $\mathbf{B} - \mathbf{A}$ is also written for $\mathbf{B} \setminus \mathbf{A}$.
- iii. If A_1, A_2, \dots , are events then so are

$$A_1 \cup A_2 \cup \dots = \bigcup_{n=1}^{\infty} \mathbf{A}_n \quad \text{and} \quad A_1 \cap A_2 \cap \dots = \bigcap_{n=1}^{\infty} \mathbf{A}_n.$$

Remarks

- i. $\overline{\Omega} = \emptyset$.
- ii. $\mathbf{A} \cup \mathbf{B} \cup \emptyset \cup \emptyset \cup \dots = \mathbf{A} \cup \mathbf{B}$.
- iii. $\mathbf{A} \cap \mathbf{B} = \Omega \setminus (\overline{\mathbf{A} \cup \mathbf{B}}) = \overline{\overline{\mathbf{A}} \cup \overline{\mathbf{B}}}$.
- iv. $\mathbf{A} \setminus \mathbf{B} = \mathbf{A} \cap \overline{\mathbf{B}}$.
- v. $\mathbf{A} \triangle \mathbf{B} := (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{B} \setminus \mathbf{A})$ (this is the definition of symmetric difference).

Definition: Let \mathbf{X} be a nonempty set. An *algebra* of sets on \mathbf{X} is a nonempty collection $A \in P(\mathbf{X})$ which is closed under finite unions and complements, i.e.,

- i. if $E_1, E_2, \dots, E_n \in A$ then $\bigcup_{k=1}^n E_k \in A$.
- ii. if $E \in A$ then $\overline{E} \in A$.

Definition: A σ -*algebra* (or σ -*field*) is an algebra which is closed under countable unions. So, a collection F of subsets of \mathbf{X} is called a σ -field if

- i. $\emptyset \in F$.
- ii. $A_1, A_2, \dots \in F \Rightarrow \bigcup_{k=1}^{\infty} A_k \in F$.
- iii. $A \in F \Rightarrow \overline{A} \in F$.

Definition: A *metric* d on a nonempty set \mathbf{X} is a function $d : \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{R}$ satisfying

- i. $d(x, y) \geq 0 \forall x, y \in \mathbf{X}$ and $d(x, y) = 0 \Rightarrow x = y$.
- ii. $d(x, y) = d(y, x) \forall x, y \in \mathbf{X}$.
- iii. $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in \mathbf{X}$ (triangle inequality).

The pair (\mathbf{X}, d) is called a *metric space*.

Definition: Let (\mathbf{X}, d) be a metric space. If $x \in \mathbf{X}$ and $r > 0$, the *open ball* of radius r about x is

$$B(r, x) = \{y \in \mathbf{X} : d(x, y) < r\}.$$

Definition: A set $E \subset \mathbf{X}$ is *open* if for every $x \in E$ there exists $r > 0$ such that $B(r, x) \subset E$.

Definition: A set is *closed* if its complement is open.

Definition: If (\mathbf{X}, d) is any metric space, the σ -field generated by the family of open sets in \mathbf{X} is called the *Borel σ -field*, denoted by $B_{\mathbf{X}}$. Its members are called Borel sets.

Special case: When $\mathbf{X} = \mathbf{R}$ (the real line), we have $B_{\mathbf{R}}$.