EE 464

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Lecture Notes Part 12b

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12.5 Mean and Covariance

Theorem: The mean of $g(X_1, X_2, \ldots, X_n)$ is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n) dx_1 \cdots dx_n.$$

Proof: Omitted.

Just like before the covariance of X_i and X_j is

$$C_{ij} = E\left[(X_i - \mu_i)(X_j - \mu_j)\right] = E[X_i X_j] - E[X_i]E[X_j]$$

and

$$\sigma_i^2 = C_{ii}.$$

Definitions: The *sample mean* is given by

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the *sample variance* is given by

$$\overline{V} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Theorem: If the random variables X_i are uncorrelated with the same mean μ and variance σ^2 then

$$E[\overline{X}] = \mu, \quad E[\overline{V}] = \sigma^2.$$

Proof: Homework problem.

Recall for random variables X_i, X_j,

$$C_{ij} = E\left[(X_i - \mu_i)(X_j - \mu_j)\right] = E[X_i X_j] - E[X_i]E[X_j].$$

This assumes X_i, X_j are real. For complex X_i, X_j,

$$C_{ij} = E\left[(X_i - \mu_i)(X_j^* - \mu_j^*)\right] = E[X_i X_j^*] - E[X_i]E[X_j^*]$$

and

$$\sigma_i^2 = C_{ii} = E[|X_i - \mu_i|^2].$$

We can construct correlation and covariance matrices, respectively, as

$$R_n = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \vdots & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix}$$

and

$$C_n = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \vdots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

where,

$$R_{ij} = E\left[X_i X_j^*\right] = E\left[(X_i^* X_j)^*\right] = \left[E(X_i^* X_j)\right]^* = R_{ji}^*,$$

$$C_{ij} = R_{ij} - \mu_i \mu_j^* = C_{ji}^*.$$

12.6 Conditional Densities

Recall

$$f(y|x) = \frac{f(x,y)}{f(x)}.$$

Similarly, for the n-dimensional case

$$f(x_n, \ldots, x_{k+1}|x_k, \ldots, x_1) = \frac{f(x_1, \ldots, x_n)}{f(x_1, \ldots, x_k)}$$

and

$$F(x_n, \ldots, x_{k+1}|x_k, \ldots, x_1) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_{k+1}} f(\alpha_n, \ldots, \alpha_{k+1}|x_k, \ldots, x_1) d\alpha_{k+1} \cdots d\alpha_n.$$

Now

$$f(x_1, x_2) = f(x_2, x_1) = f(x_2|x_1) f(x_1),$$

$$f(x_1, x_2, x_3) = f(x_3|x_2, x_1) f(x_2, x_1) = f(x_3|x_2, x_1) f(x_2|x_1) f(x_1).$$

Generalizing, we have the chain rule

$$f(x_1, \ldots, x_n) = f(x_n|x_{n-1}, \ldots, x_1) \cdots f(x_2|x_1) f(x_1).$$

Let $n = 3$. Then

$$f(x_1|x_3) = \int_{-\infty}^{\infty} f(x_1, x_2|x_3) dx_2 = \int_{-\infty}^{\infty} f(x_1|x_2, x_3) f(x_2|x_3) dx_2.$$

We also have the concept of conditional expected values.

$$E[X_1|x_2, \ldots, x_n] = \int_{-\infty}^{\infty} x_1 f(x_1|x_2, \ldots, x_n) dx_1.$$

In the above $x_2, \ldots, x_n$ are fixed constants. Thus, $E[X_1|x_2, \ldots, x_n]$ is a constant. Therefore, $E[X_1|X_2, \ldots, X_n]$ is a random variable. We can compute the expected value of this random variable as

$$E\left[E(X_1|X_2, \ldots, X_n)\right] = E(X_1).$$

The proof of this is similar to our proof of

$$E\left[E(X|Y)\right] = E(X).$$