EE 464

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Lecture Notes Part 12a

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12.0 Sequences of Random Variables

Here we will deal with n-dimensional versions of some concepts previously studied plus some additional concepts.

12.1 Introduction

Definition: A *random vector* is denoted

$$\underline{X} = [X_1, X_2, \ldots, X_n]$$

where each $X_i,\ i = 1, 2, \ldots, n$ is a random variable.

Given a region D in n-dimensional space, the probability that $\underline{X}$ is in D is given by

$$P(\underline{X} \in D) = \int \cdots \int_D f(\underline{x}) d\underline{x}, \quad \underline{x} = [x_1, x_2, \ldots, x_n].$$

Definition: The *joint distribution* of the random vector $\underline{\mathrm{X}}$ is

$$F(\underline{x}) = F(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)$$

and the *joint density* is

$$f(\underline{x}) = f(x_1, x_2, \ldots, x_n) = \frac{\partial^n F(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}.$$

Note that integration of $f(x_1, x_2, \ldots, x_n)$ with respect to any of the variables gives the joint density of the remaining variables. For example, if $n = 4$,

$$f(x_1, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3, x_4) dx_2 dx_4.$$

12.2 Transformation of a Random Vector

Given $k = n$ functions (if $k < n$ we can use auxillary random variables to make $k = n$ just like we did for $n = 2$ dimensions)

$$g_1(\underline{x}),\ g_2(\underline{x}), \ldots,\ g_n(\underline{x})$$

we form

$$Y_1 = g_1(\underline{X}),\ Y_2 = g_2(\underline{X}), \ldots,\ Y_n = g_n(\underline{X})$$

and solve the system

$$y_1 = g_1(\underline{x}),\ y_2 = g_2(\underline{x}), \ldots,\ y_n = g_n(\underline{x}) \text{ for } \underline{x} = [x_1, x_2, \ldots, x_n].$$

If we have a single solution then

$$f_{\underline{Y}}(y_1, y_2, \ldots, y_n) = \frac{f_{\underline{X}}(x_1, x_2, \ldots, x_n)}{|J(x_1, x_2, \ldots, x_n)|}$$

where

$$J(x_1, x_2, \ldots, x_n) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}^{-1}$$

If we have several solutions then we add the corresponding terms as was stated for the 2-dimensional case.

12.3 Independence

Definition: The random variables $X_1, \ldots, X_n$ are called *independent* if each of the events

$$\{\omega : X_1(\omega) \le x_1\}, \ldots, \{\omega : X_n(\omega) \le x_n\}$$

are independent.

So $X_1, \ldots, X_n$ independent implies

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\cdots f(x_n),$$

$$f(x_2, x_5) = f(x_2)f(x_5),$$

etc.

Similarly, $X_1, \ldots, X_n$ independent implies

$$F(x_1, x_2, \ldots, x_n) = F(x_1)F(x_2)\cdots F(x_n),$$

$$F(x_2, x_5) = F(x_2)F(x_5),$$

etc.

Furthermore, $X_1, \ldots, X_n$ independent implies $g_1(X_1), \ldots, g_n(X_n)$ are independent.

In a combined experiment, where we repeat the same underlying experiment over and over independently, then each outcome of the experiment will have the same distribution. In this case we say the random variables formed for each experiment are *independent and identically distributed* (i.i.d).

12.4 Order Statistics

Here we are concerned with an ordered sequence of random variables. Let $X_1, \ldots, X_n$ denote n i.i.d. random variables each with cdf $F_X(x)$. Then $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the ordered version of $X_1, \ldots, X_n$, i.e.,

$$X_{(1)} = \min\{X_1, \ldots, X_n\}$$

$$X_{(n)} = \max\{X_1, \ldots, X_n\}$$

and

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$

Notation: In this section, to simplify notation, we will write

$$Y_1 = X_{(1)},\ Y_2 = X_{(2)}, \ldots,\ Y_n = X_{(n)}.$$

Theorem: Let $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ represent the order statistics from a cdf $F_X(x)$. Then the marginal cdf of Y_k, $k \in \{1, 2, \ldots, n\}$ is

$$F_{Y_k}(y) = \sum_{j=k}^{n} \binom{n}{j} [F_X(y)]^j [1 - F_X(y)]^{n-j}.$$

Proof: Define the indicator function I_L by

$$I_L(x) = \begin{cases} 1, & x \in L, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $y \in \mathbf{R}$ be fixed and let $Z_i = I_{(-\infty,\ y]}(x_i)$. Then

$$\sum_{i=1}^{n} Z_i = \text{number of } X_i \leq y.$$

This implies

$$\sum_{i=1}^{n} Z_i \sim B(n, F_X(y)),$$

i.e.,

$$P\left(\sum_{i=1}^{n} Z_i = \beta\right) = \binom{n}{\beta}[F_X(y)]^{\beta}[1 - F_X(y)]^{n-\beta}.$$

Now

$$F_{Y_k}(y) = P(Y_k \le y)$$

so if $Y_k \le y$ then the number of $X_i \ge y$ is $\ge k$ and conversely. Thus,

$$F_{Y_k}(y) = P(Y_k \le y) = P\left(\sum_{i=1}^{n} Z_i \ge k\right)$$

$$= \sum_{j=k}^{n} \binom{n}{j} [F_X(y)]^{j} [1 - F_X(y)]^{n-j}.$$

Corollary:

$$F_{Y_n}(y) = [F_X(y)]^n$$

and

$$F_{Y_1}(y) = \sum_{j=1}^{n} \binom{n}{j} [F_X(y)]^{j} [1 - F_X(y)]^{n-j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} [F_X(y)]^{j} [1 - F_X(y)]^{n-j} - [1 - F_X(y)]^n$$

which implies

$$F_{Y_1}(y) = 1 - [1 - F_X(y)]^n.$$

Theorem: If the X_i are continuous random variables then the density of Y_k (the kth-order statistic) is

$$f_{Y_k}(y) = \frac{n!}{(k-1)!\,(n-k)!} [F_X(y)]^{k-1} [1 - F_X(y)]^{n-k} f_X(y).$$

Proof:

$$f_{Y_k}(y) = \lim_{\Delta y \to 0} \frac{F_{Y_k}(y + \Delta y) - F_{Y_k}(y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{P(y < Y_k \le y + \Delta y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{P\left[(k-1) \text{ of the } X_i \leq y, \text{ one } X_i \in (y, y+\Delta y], (n-k) \text{ of the } X_i > y+\Delta y\right]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{[F_X(y)]^{k-1} [F_X(y+\Delta y) - F_X(y)] [1 - F_X(y+\Delta y)]^{n-k}}{\Delta y} \frac{n!}{(k-1)!\ 1!\ (n-k)!} \right]$$

$$= \frac{n!}{(k-1)!\ (n-k)!} [F_X(y)]^{k-1} [1 - F_X(y)]^{n-k} f_X(y).$$

Example: Say $X_i,\ i = 1, 2, \ldots, n$ is exponentially distributed with parameter λ. Then

$$f_{X_i}(x_i) = f_X(x) = \lambda e^{-\lambda x} U(x).$$

Note that

$$E(X_i) = E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

Let $Y_1 = \min\{X_1, \ldots, X_n\}$. Then

$$f_{Y_1}(y) = \frac{n!}{(1-1)!\ (n-1)!} [F_X(y)]^{1-1} [1 - F_X(y)]^{n-1} f_X(y)$$

$$= n [1 - F_X(y)]^{n-1} f_X(y).$$

Now

$$F_X(y) = \int_0^y \lambda e^{-\lambda x} dx = \left(1 - e^{-\lambda y}\right) U(y)$$

which implies

$$f_{Y_1}(y) = n \left[e^{-\lambda y}\right]^{n-1} \lambda e^{-\lambda y} U(y)$$

$$= (n\lambda) e^{-(n\lambda) y} U(y).$$

So Y_1 is exponentially distributed with parameter $(n\lambda)$ which implies

$$E(Y_1) = \frac{1}{n\lambda}.$$

Example: Let X be exponentially distributed with parameter λ. Then

$$f_X(x) = \lambda e^{-\lambda x} U(x).$$

Let $Y = cX,\ c > 0$. Then

$$F_Y(y) = P(Y \leq y) = P(cX \leq y) = P(X \leq y/c) = 1 - e^{-\lambda y/c}.$$

Thus

$$f_Y(y) = \frac{\lambda}{c} e^{-\frac{\lambda}{c} y}$$

which implies Y is exponentially distributed with parameter (λ/c). Now let $c = 1/n$. Then Y is exponentially distributed with parameter $(n\lambda)$ which is the same as $Y_1 = \min\{X_1, \ldots, X_n\}$ with each X_i exponentially distributed with parameter λ.