EE 464

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Lecture Notes Part 11b

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11.2 Joint Characteristic Functions

Definition: The *joint characteristic function* of the random variables X and Y is

$$\phi_{XY}(\omega_1, \omega_2) = E\left[e^{i(\omega_1 X + \omega_2 Y)}\right].$$

So,

$$\phi_{XY} : \mathbf{R}^2 \to \mathbf{C}^2.$$

For X, Y discrete we have

$$\phi_{XY}(\omega_1, \omega_2) = \sum_{k,l} e^{i(\omega_1 x_k + \omega_2 y_l)} P(X = x_k, Y = y_l).$$

For X, Y continuous we have

$$\phi_{XY}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i(\omega_1 x + \omega_2 y)} dx dy.$$

Using the inversion formula for the 2-dimensional Fourier transform (with a sign change) we get

$$f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{XY}(\omega_1, \omega_2) e^{-i(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2.$$

Definitions: The *marginal characteristic functions* of the random variables X and Y are

$$\phi_X(\omega) = E\left(e^{i\omega X}\right) \text{ and } \phi_Y(\omega) = E\left(e^{i\omega Y}\right).$$

Note that $\phi_X(\omega) = \phi_{XY}(\omega, 0)$ and $\phi_Y(\omega) = \phi_{XY}(0, \omega)$.

Claim: X and Y are independent if and only if

$$\phi_{XY}(\omega_1, \omega_2) = \phi_X(\omega_1)\phi_Y(\omega_2).$$

Proof:
"$\Longrightarrow$" (Necessity or "only if" part).

Assume that X and Y are independent. Then

$$\phi_{XY}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i(\omega_1 x + \omega_2 y)} dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_X(x) f_Y(y) e^{i\omega_1 x} e^{i\omega_2 y} dx dy$$

$$= \int_{-\infty}^{\infty} f_X(x) e^{i\omega_1 x} dx \int_{-\infty}^{\infty} f_Y(y) e^{i\omega_2 y} dx$$

$$= E\left[e^{i\omega_1 x}\right] E\left[e^{i\omega_2 y}\right] = \phi_X(\omega_1)\phi_Y(\omega_2).$$

"$\Longleftarrow$" (Sufficiency or "if" part).

Assume $\phi_{XY}(\omega_1, \omega_2) = \phi_X(\omega_1)\phi_Y(\omega_2)$. Then

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{XY}(\omega_1, \omega_2) e^{-i(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_X(\omega_1)\phi_Y(\omega_2) e^{-i\omega_1 x} e^{-i\omega_2 y} d\omega_1 d\omega_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega_1) e^{-i\omega_1 x} d\omega_1 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_Y(\omega_2) e^{-i\omega_2 y} d\omega_2$$

$$= f_X(x) f_Y(y) \Rightarrow \; X \text{ and } Y \text{ are independent.}$$

Theorem: If the random variables X and Y are independent and $Z = X+Y$ then

$$\phi_Z(\omega) = \phi_X(\omega)\phi_Y(\omega).$$

Proof:

$$\phi_Z(\omega) = E\left[e^{i\omega Z}\right] = E\left[e^{i\omega(X+Y)}\right] = E\left[e^{i\omega X} e^{i\omega Y}\right]$$

$$= E\left[e^{i\omega X}\right] E\left[e^{i\omega Y}\right] = \phi_X(\omega)\phi_Y(\omega).$$

Recall for X and Y independent that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \quad \text{(convolution)}.$$

Theorem: Let $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ and let X and Y be independent. Then $Z = X + Y$ is also normal and

$$E(Z) = \mu_X + \mu_Y, \quad Var(Z) = Var(X) + Var(Y).$$

Proof:

$$\phi_X(\omega) = E\left(e^{i\omega X}\right) = \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} e^{i\omega x} e^{-(x-\mu_X)^2/2\sigma_X^2} dx.$$

Recall

$$M_X(s) = E\left(e^{sX}\right) = e^{s\mu_X + \frac{1}{2}\sigma_X^2 s^2}.$$

So

$$M_X(i\omega) = E\left(e^{i\omega X}\right) = \phi_X(\omega)$$

when $M_X(s)$ exists (recall $\phi_X(\omega)$ always exists). Thus,

$$\phi_X(\omega) = e^{i\mu_X\omega - \frac{1}{2}\sigma_X^2\omega^2}.$$

Similarly,

$$\phi_Y(\omega) = e^{i\mu_Y\omega - \frac{1}{2}\sigma_Y^2\omega^2}.$$

Therefore,

$$\phi_Z(\omega) = \phi_X(\omega)\phi_Y(\omega) = e^{i(\mu_X+\mu_Y)\omega - \frac{1}{2}(\sigma_X^2+\sigma_Y^2)\omega^2}.$$

Hence,

$$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

More generally, if we have the above conditions and $Z = aX + bY$, then

$$\phi_{aX}(\omega) = e^{ia\mu_X\omega - \frac{1}{2}a^2\sigma_X^2\omega^2} = \phi_X(a\omega),$$

$$\phi_{bX}(\omega) = e^{ib\mu_X\omega - \frac{1}{2}b^2\sigma_X^2\omega^2} = \phi_X(b\omega),$$

and

$$\phi_Z(\omega) = \phi_{aX}(\omega)\phi_{bY}(\omega).$$

Thus,

$$Z \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2).$$

11.3 Conditional Distributions

Consider

$$F_Y(y|X \le x) = P(Y \le y|X \le x) = \frac{P(X \le x,\ Y \le y)}{P(X \le x)} = \frac{F_{XY}(x,y)}{F_X(x)}.$$

Thus,

$$f_Y(y|X \le x) = \frac{d}{dy}F_Y(y|X \le x) = \frac{\frac{\partial F_{XY}(x,y)}{\partial y}}{F_X(x)}.$$

Now

$$F_{XY}(x,y|x_1<X\le x_2)=P(X\le x,\ Y\le y\ |x_1<X\le x_2)$$
$$=\frac{P(X\le x,\ Y\le y,\ x_1<X\le x_2)}{P(x_1<X\le x_2)}. \qquad (\star)$$

If $x>x_2$,

$$(\star)=\frac{P(x_1<X\le x_2,\ Y\le y)}{P(x_1<X\le x_2)}=\frac{F_{XY}(x_2,y)-F_{XY}(x_1,y)}{F_X(x_2)-F_X(x_1)}.$$

If $x_1<x\le x_2)$,

$$(\star)=\frac{P(x_1<X\le x,\ Y\le y)}{P(x_1<X\le x_2)}=\frac{F_{XY}(x,y)-F_{XY}(x_1,y)}{F_X(x_2)-F_X(x_1)}.$$

If $x<x_1$,

$$(\star)=0.$$

Now

$$f_{XY}(x,y)=\frac{\partial^2 F_{XY}(x,y)}{\partial x\partial y}$$

so

$$f_{XY}(x,y|x_1<X\le x_2)=\begin{cases}\dfrac{f_{XY}(x,y)}{F_X(x_2)-F_X(x_1)}, & x_1<x\le x_2,\\[2ex] 0, & \text{elsewhere.}\end{cases}$$

Consider next

$$F_Y(y|x_1<X\le x_2)=P(Y\le y|x_1<X\le x_2)=\frac{P(x_1<X\le x_2,\ Y\le y)}{P(x_1<X\le x_2)}$$
$$=\frac{F_{XY}(x_2,y)-F_{XY}(x_1,y)}{F_X(x_2)-F_X(x_1)}.$$

Thus

$$f_Y(y|x_1<X\le x_2)=\frac{\partial}{\partial y}\frac{\int_{-\infty}^{x_2}\int_{-\infty}^{y}f_{XY}(x,\theta)d\theta dx-\int_{-\infty}^{x_1}\int_{-\infty}^{y}f_{XY}(x,\theta)d\theta dx}{F_X(x_2)-F_X(x_1)}$$
$$=\frac{\int_{x_1}^{x_2}f_{XY}(x,y)dx}{F_X(x_2)-F_X(x_1)}.$$

Now let $x_1 = x,\ x_2 = x + \Delta x$. Then

$$f_Y(y|x < X \leq x + \Delta x) = \frac{\displaystyle\int_x^{x+\Delta x} f_{XY}(\alpha, y)d\alpha}{F_X(x + \Delta x) - F_X(x)}.$$

Now let $\Delta x \to 0$ to get

$$\frac{\displaystyle\int_x^{x+\Delta x} f_{XY}(\alpha, y)d\alpha}{F_X(x + \Delta x) - F_X(x)} \longrightarrow \frac{f_{XY}(x, y)\Delta x}{f_X(x)\Delta x}.$$

Thus,

$$f_Y(y|X = x) = \lim_{\Delta x \to 0} f_Y(y|x < X \leq x + \Delta x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

Notation:

$$f(y|x) = f_Y(y|x) = f_Y(y|X = x), \quad f(x|y) = f_X(x|y) = f_X(x|Y = y),$$

$$f(y|x) = \frac{f(x, y)}{f(x)}, \quad f(x|y) = \frac{f(x, y)}{f(y)}.$$

Compare to

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \ \text{ for events } A \text{ and } B,\ P(B) \neq 0.$$

Notation: If X and Y are independent

$$f_Y(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y)$$

and similarly $f_X(x|y) = f_X(x)$.

Now

$$f_X(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{f_Y(y|x)f_X(x)}{f_Y(y)}.$$

Also

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

and

$$f_{XY}(x,y) = f_Y(y|x)f_X(x).$$

Hence,

$$f_Y(y) = \int_{-\infty}^{\infty} f_Y(y|x)f_X(x)dx.$$

This is total probability. We thus get Bayes' theorem for densities

$$f_X(x|y) = \frac{f_Y(y|x)f_X(x)}{\displaystyle\int_{-\infty}^{\infty} f_Y(y|x)f_X(x)dx}.$$

Discrete Type: Let $p_i = P(X = x_i),\ p_{ik} = P(X = x_i,\ Y = y_k)$. Then.

$$P(Y = y_k|X = x_i) = \frac{P(X = x_i,\ Y = y_k)}{P(X = x_i)} = \frac{p_{ik}}{p_i}.$$