EE 464

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Lecture Notes Part 10b

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10.2 Independence

Definition: Let (X, Y) be a two-dimensional discrete random variable. We say X and Y are *independent* if $p_{ik} = p_i p_k$, where $p_{ik} = P(X = x_i, Y = y_k)$, $p_i = P(X = x_i)$, $p_k = P(Y = y_k)$, i.e., $P(X = x_1, Y = y_k) = P(X = x_i)P(Y = y_k)\ \forall\ i, k$.

Definition: Let (X, Y) be a two-dimensional continuous random variable. We say X and Y are *independent* if $f(x, y) = f_X(x)f_Y(y)$, or equivalently $F(x, y) = F_X(x)F_Y(y)$, or equivalently $P(X \in S_X, Y \in S_Y) = P(X \in S_X)P(Y \in S_Y)$, where S_X and S_Y are arbitrary measurable sets on the x-axis and y-axis, respectively.

Example: From before

$$f(x, y) = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

We found

$$f_X(x) = \begin{cases} 2x^2 + \dfrac{2x}{3}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{3} + \dfrac{y}{6}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Here $f(x, y) \neq f_X(x)f_Y(y)$ so X and Y are not independent.

Theorem: Let (X, Y) be a two-dimensional random variable. Let A and B be events whose occurrence (or nonoccurrence) depends only on X and Y, respectively. (That is, $A \subset R_X,\ B \subset R_Y$.) Then, if X and Y are independent random variables, $P(A \cap B) = P(A)P(B)$.

Proof: (continuous case only).

$$P(A \cap B) = \int\int_{A \cap B} f(x, y)dxdy$$

10.3 One Function of Two Random Variables

Given random variables X and Y and a function $g(x, y)$, we form the random variable

$$Z = g(X, Y).$$

We find the distribution function as

$$F_Z(z) = P(Z \leq z) = P(g(X, Y) \leq z) = P((x, y) \in D_z)$$

where, D_z is the region in the xy-plane consisting of the set

$$\{(x, y) : x \in R_X,\ y \in R_Y,\ g(x, y) \leq z\}.$$

So,

$$F_Z(z) = \int\int_{D_z} f(x, y)dxdy$$

where, $f(x, y)$ is the joint density of X and Y.

Example: Given X and Y with joint *pdf*

$$f(x, y) = \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Are X and Y independent?

b. Find the distribution function and the *pdf* of $Z = \min\{X, Y\}$.

Solution:

a. X and Y independent if and only if $f(x, y) = f_X(x)f_Y(y)$. Now

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{\infty} e^{-(x+y)}dy$$

or

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} e^{-y}, & y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

So $f(x, y) = e^{-(x+y)} = e^{-x}e^{-y} = f_X(x)f_Y(y) \Rightarrow X$ and Y are independent.

b. $Z = \min\{X, Y\}$. Now

$$F_Z(z) = P(Z \le z) = P(\min\{X, Y\} \le z)$$
$$= 1 - P(\min\{X, Y\} > z)$$
$$= 1 - P(X > z, Y > z) = 1 - P(X > z)P(Y > z).$$

Now

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_{o}^{x} e^{-u}du = \left(1 - e^{-x}\right)U(x).$$

Similarly,

$$F_Y(y) = \left(1 - e^{-y}\right)U(y).$$

So

$$F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z)) = \left(1 - e^{-z}e^{-z}\right)U(z)$$

or

$$F_Z(z) = \begin{cases} 1 - e^{-2z}, & z \ge 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, $f_Z(z) = \frac{d}{dz}F_Z(z)$ or

$$f_Z(z) = \begin{cases} 2e^{-2z}, & z \ge 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Example: Suppose that X and Y are independent random variables each uniformly distributed over $[0, 1]$. Let $Z = X + Y$. Find the *pdf* of Z, $f_Z(z)$, and the distribution function, $F_Z(z)$.

Solution: The joint *pdf* is

$$f(x, y) = f_X(x)f_Y(y),$$

where

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Now,
$$F_Z(z) = P(Z \le z) = P(X + Y \le z) = \int\int_{(x,y):x+y\le z} f(x,y)dxdy.$$
There are four cases to consider:

1. $z < 0$. Set $x + y = z \Rightarrow y = -x + z$. Here $F_Z(z) = 0$.
2. $0 \le z \le 1$.
$$F_Z(z) = \int_0^z \int_0^{-x+z} dydx = z^2/2.$$
3. $1 < z \le 2$.
$$F_Z(z) = \int_0^{z-1}\int_0^1 dydx + \int_{z-1}^1 \int_0^{-x+z} dydx = 2z - z^2/2 - 1.$$
4. $z > 2$. Here $F_Z(z) = 1$.

So
$$F_Z(z) = \begin{cases} 0, & z < 0, \\ z^2/2, & 0 \le z \le 1, \\ 2z - z^2/2 - 1, & 1 < z \le 2, \\ 1, & z > 2. \end{cases}$$
Definition: Two random variables X and Y are said to be *jointly normal* if
$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}}$$
$$\cdot \exp\left[\frac{-1}{2(1-r^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - 2r\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right].$$
Here $|r| < 1$ where r is called the correlation coefficient (more about this later). Integration shows
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \cdot \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$
and
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \cdot \exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right).$$

If $r = 0$ then X and Y are said to be uncorrelated and in the normal case they are also independent.

Example: Let

$$f(x,y) = \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right).$$

Here $f(x,y) = f_X(x)f_Y(y)$ where

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{y^2}{2\sigma^2}\right).$$

Find $f_Z(z)$ where $Z = \sqrt{X^2+Y^2}$.

Solution: Here $Z^2 = X^2 + Y^2$. Let

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dxdy = rdrd\theta.$$

Then,

$$F_Z(z) = \int\int_{D_z} f_{XY}(x,y)dxdy = \int\int_{D_z} \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

$$= \int\int_{D_z^*} f_{R\Theta}(r,\theta)drd\theta$$

where

$$D_z = \left\{(x,y) : \sqrt{x^2+y^2} \le z\right\}, \quad D_z^* = \{r,\theta) : 0 < r \le z, 0 < \theta \le 2\pi\}.$$

So

$$F_Z(z) = \frac{1}{2\pi\sigma^2}\int_0^z \int_0^{2\pi} re^{-r^2/2\sigma^2} d\theta dr$$

$$= 1 - e^{-z^2/2\sigma^2}, \; z \ge 0.$$

Thus,

$$f_Z(z) = \begin{cases} \dfrac{z}{\sigma^2} \cdot e^{-z^2/2\sigma^2}, & z \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

This is a *Rayleigh* density.

10.4 Two Functions of Two Random Variables

Here we have two random variables X and Y and form

$$Z = g(X, Y),\ W = h(X, Y).$$

Now

$$F_{ZW}(z, w) = P(Z \leq z, W \leq w) = P(g(X, Y) \leq z, h(X, Y) \leq w)$$

$$= P((X, Y) \in D_{zw})$$

where, D_{zw} is the region in the xy-plane consisting of the set

$$\{(x, y) : x \in R_X,\ y \in R_Y,\ g(x, y) \leq z,\ h(x, y) \leq w\}.$$

So,

$$F_{ZW}(z, w) = \int\int_{D_{zw}} f_{XY}(x, y) dx dy.$$

Theorem: Let X and Y be two random variables and let $Z = g(X, Y)$ and $W = h(X, Y)$. Let $g(x, y) = z$ and $h(x, y) = w$ have solutions $(x_n, y_n), n = 1, 2, \ldots$. Then

$$f_{ZW}(z, w) = \frac{f_{XY}(x_1, y_1)}{|J(x_1, y_1)|} + \frac{f_{XY}(x_2, y_2)}{|J(x_2, y_2)|} + \cdots$$

where J is the Jacobian defined as

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1}$$

Proof: See posted handout.

Example: Suppose we throw a dart at a unit circle. Let (x, y) denote the coordinates where it lands. The area of the unit circle is $\pi r^2 = \pi$ so assume

$$f_{XY}(x,y) = \begin{cases} 1/\pi, & (x,y) \text{ inside circle,} \\ 0, & \text{elsewhere.} \end{cases}$$

Define $R = \sqrt{X^2 + Y^2}$. Find $f_R(r)$.

Solution: Let $\Phi = \tan^{-1}(Y/X)$. Then

$$R = g(X,Y), \text{ where } g(x,y) = \sqrt{x^2 + y^2}$$

and

$$\Phi = h(X,Y), \text{ where } h(x,y) = \tan^{-1}(y/x).$$

We solve for x, y as

$$x = r\cos\phi, \quad y = r\sin\phi.$$

Thus,

$$J(r,\phi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix}^{-1} = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix}^{-1} = \frac{1}{r}.$$

So,

$$f_{R\Phi}(r,\phi) = f_{XY}(r\cos\phi, r\sin\phi)r = r/\pi, \ 0 \le r \le 1, \ 0 \le \phi < 2\pi$$

and

$$f_R(r) = \int_0^{2\pi} f_{R\Phi}(r,\phi)d\phi = \int_0^{2\pi} \frac{r}{\pi} d\phi = 2r, \ 0 \le r \le 1.$$

Theorem: Let (X, Y) be a continuous two-dimensional random variable and assume X and Y are independent. Let $W = XY$. Then

$$f_W(w) = \int_{-\infty}^{\infty} f_X(z) f_Y(w/z) \left|\frac{1}{z}\right| dz.$$

Proof: $w = xy$. Let $z = x$. Thus, $x = z$ and $y = w/z$.

$$J(z,w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1} = \begin{vmatrix} 1 & 0 \\ -w/z^2 & 1/z \end{vmatrix}^{-1} = z.$$

So,

$$f_{ZW}(z,w) = f_{XY}(z, w/z)\left|\frac{1}{z}\right| = f_X(z) f_Y(w/z)\left|\frac{1}{z}\right|.$$

Thus,

$$f_W(w) = \int_{-\infty}^{\infty} f_X(z) f_Y(w/z)\left|\frac{1}{z}\right| dz.$$

Theorem: Let (X, Y) be a continuous two-dimensional random variable and assume X and Y are independent. Let $Z = X/Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(wz) f_Y(w)\,|w|\,dw.$$

Proof: $z = x/y$. Let $w = y$. Thus, $x = wz$ and $y = w$.

$$J(z,w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix}^{-1} = \frac{1}{w}.$$

So,

$$f_{ZW}(z,w) = f_X(wz) f_Y(w)\,|w|\,.$$

Thus,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(wz) f_Y(w)\,|w|\,dw.$$

Theorem: Let (X, Y) be a continuous two-dimensional random variable. Let $Z = X + Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z - w) dw.$$

Proof: $z = x + y$. Let $w = x$. Thus, $y = z - w$ and $x = w$.

$$J(z,w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}^{-1} = -1.$$

So,

$$f_{ZW}(z,w) = f_{XY}(w, z - w).$$

Thus,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z - w) dw.$$

Special Case: If X and Y are independent then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)dw = f_X * f_Y.$$

We can deduce this last result a more direct way as follows: Recall X and Y are independent and $Z = X + Y$.

$$F_Z(z) = P(Z \leq z) = P((X+Y) \leq z) = \int\int_{D_z} f_X(x) f_Y(y) dx dy$$

where

$$D_z = \{(x,y) : x + y \leq z\}.$$

Thus

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy dx \\ &= \int_{-\infty}^{\infty} f_X(x) \left[\int_{-\infty}^{z-x} f_Y(y) dy \right] dx. \end{aligned}$$

Then

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

Example: Suppose T_1 and T_2 represent the life lengths of electronic components with *pdf*

$$f_{T_1}(t_1) = \begin{cases} \alpha_1 e^{-\alpha_1 t_1}, & t_1 \geq 0, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_{T_2}(t_2) = \begin{cases} \alpha_2 e^{-\alpha_2 t_2}, & t_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Suppose component 2 turns on when component 1 ceases operation. Find the *pdf* of the total life length $T = T_1 + T_2$.

Solution:

i. Assume $\alpha_1 \neq \alpha_2$. Then

$$f_T(t) = \int_{-\infty}^{\infty} f_{T_1}(t_1) f_{T_2}(t - t_1) dt_1, \ t \geq 0.$$

Now $t_1 \geq 0$ (due to f_{T_1}) and $(t - t_1) \geq 0$ (due to f_{T_2}). Thus $0 \leq t_1 \leq t$. So,

$$f_T(t) = \int_0^t \alpha_1 e^{-\alpha_1 t_1} \alpha_2 e^{-\alpha_2 (t-t_1)} dt_1$$

or

$$f_T(t) = \begin{cases} \dfrac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} \left(e^{-\alpha_1 t_1} - e^{-\alpha_2 t_2} \right), & t \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

ii. Assume $\alpha_1 = \alpha_2 = \alpha$. Then

$$f_T(t) = \int_0^t \alpha e^{-\alpha t_1} \alpha e^{-\alpha (t-t_1)} dt_1$$

or

$$f_T(t) = \begin{cases} \alpha^2 t e^{-\alpha t}, & t \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Linear Transformations

Let $Z = aX + bY$, $W = cX + dY$ for constants a, b, c, d. Then the values these random variables can take on can be written as

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we let M be the 2×2 matrix above then if M^{-1} exists, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} z \\ w \end{bmatrix}.$$

Then $x = Az + Bw$, $y = Cz + Dw$ for some constants A, B, C, D. Now

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

So,

$$f_{ZW}(z, w) = \frac{1}{|ad - bc|} f_{XY} \left(Az + Bw, Cz + Dw \right).$$

Theorem: Let X and Y be two independent random variables each of which may assume only nonnegative integral values. Let

$$p_k = P(X = k), k = 0, 1, 2, \ldots$$

and

$$q_r = P(Y = r), r = 0, 1, 2, \ldots.$$

Let $W = X + Y$ and let $w_i = P(W = i)$. Then

$$w_i = \sum_{k=0}^{i} p_k q_{i-k}, \ i = 0, 1, 2 \ldots.$$

Proof: $w_i = P(w = i)$ so

$$w_i = P(X = 0, Y = i \text{ or } X = 1, Y = i - 1 \text{ or } \cdots \text{ or } X = i, Y = 0)$$

$$= \sum_{k=0}^{i} P(X = k, Y = i - k) = \sum_{k=0}^{i} P(X = k) P(Y = i - k) = \sum_{k=0}^{i} p_k q_{i-k}.$$